Belief-Kinematics Jeffrey's Rules in the Theory of Evidence

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Abstract

This paper studies the problem of revising beliefs using uncertain evidence in a framework where beliefs are represented by a belief function. We introduce two *new* Jeffrey's rules for the revision based on two forms of *belief kinematics*, an evidence-theoretic counterpart of probability kinematics. Furthermore, we provide two distance measures for belief functions and show that the two belief kinematics are optimal in the sense that they minimize their corresponding distance measures.

1 INTRODUCTION

Reasoning about uncertainty is a fundamental issue for Artificial Intelligence [HALPERN, 2005]. Numerous approaches have been proposed, including Dempster-Shafer theory of belief functions [SHAFER, 1976] (also called the theory of evidence or simply DS theory). Ever since the pioneering works by Dempster and Shafer, the theory of belief functions has become a powerful formalism in Artificial Intelligence for knowledge representation and decision-making.

In this paper, we study the revision of beliefs using uncertain evidence and we represent beliefs as belief functions. Our main contribution is to introduce two *new* Jeffrey's rules for the revision based on two different forms of *belief kinematics*, an evidence-theoretic counterpart of probability kinematics [JEFFREY, 1983]. The first rule, called *inner* revision, generalizes the geometric conditionalization rule, and the other, *outer revision*, generalizes the Dempster rule of conditioning. These two Jeffrey's rules specify uncertain evidence in terms of the *effect* it has on beliefs once accepted, and the specification is actually a function of both evidence strength and beliefs held prior to obtaining evidence. Once new evidence is accepted, a prior belief function bel on a frame Ω of discernment is revised to a new posterior belief function bel' on the same frame. This method requires us to specify *uncertain* evidence by providing a belief function bel_e on a coarser frame with less distinctions of the attention. This coarser frame is actually based on a partition of the frame Ω and hence is represented as a subalgebra $\mathcal{B}$ of the powerset 2^{Ω} of Ω whose atoms forming a partition of Ω.

The principle of belief kinematics on $\mathcal{B}$ says that, although the prior belief function bel and the posterior one bel' may disagree on propositions in $\mathcal{B}$, they agree on their *relevance* to all propositions in 2^{Ω}. Providing a reasonable representation of the notion of relevance in belief kinematics is the main challenge in this paper. For each of the two new Jeffrey's rules, we formalize a form of belief kinematics and characterize relevance in this belief kinematics by a *conditional belief function* on Ω with respect to the coarsening frame $\langle \Omega, \mathcal{B} \rangle$. Our definition of conditional belief functions differs from Dempster's rule of conditioning [SHAFER, 1976] in that our definition depends on the coarsening frame $\langle \Omega, \mathcal{B} \rangle$ while Dempster's rule doesn't. Our conditional belief functions are natural generalizations of classical conditional probability functions and provide a measure of the relevance of any proposition in $\mathcal{B}$ to all propositions in 2^{Ω}.

In this paper, we incorporate the above principle of belief kinematics into the two *new* Jeffrey's rules in the theory of belief functions by satisfying the following constraints:

- (Constraint 1) These two rules should be a natural generalization of Jeffrey's rule in probability theory, i.e., they should be the same as Jeffrey's rule in probability theory when the prior belief function bel is a probability function.
- (Constraint 2) On the coarsening frame $\langle \Omega, \mathcal{B} \rangle$, the posterior belief function bel' according to the rules should agree with the belief function bel_e that specifies the evidence.
- (Constraint 3) The revision rules should obey some natural evidence-theoretic generalization of probabil-

ity kinematics like the above belief kinematics.

Any Jeffrey's rule in DS theory should at least meet Constraint 1. Constraint 2 is Smets' distinguishing constraint **C1** [SMETS, 1993A]. Constraint 3 is the most important one and is the key point of this paper. We believe that this constraint for Jeffrey's rule in the theory of evidence is as important as the principle of probability kinematics is for Jeffrey's rule in probability theory. Unlike similar rules for belief functions found in the literature (See Section 5), our new Jeffery's rules *naturally* transfer important properties of probabilistic belief revisions to the theory of evidence.

In order to show that the above revisions based on belief kinematics are optimal, we provide for each revision rule a distance measure for bounding belief changes due to the revisions and show that the belief function obtained according to the corresponding form of belief kinematics is the closest to the prior one among all belief functions satisfying Constraint 2.

2 JEFFREY'S RULE IN PROBABILITY THEORY

Let Pr be a probability function on a probability space $\langle \Omega, \mathcal{A} \rangle$ where $\mathcal{A}$ is the Boolean algebra of subsets of Ω with the usual set operations. Suppose that new evidence suggests the desirability of revising Pr and that the total evidence determines a family $\mathcal{E}$ of mutually exclusive and exhaustive subsets of Ω and a probability function Pr_e on the Boolean algebra $\mathcal{B}$ of finite unions of elements of $\mathcal{E}$. Without loss of generality, we assume that $Pr_e(E) > 0$ for all $E \in \mathcal{E}$. The new posterior probability function Pr' on $\langle \Omega, \mathcal{A} \rangle$ proposed by *Jeffrey's rule* is as follows: for any $A \subseteq \Omega$,

$$Pr'(A) = \sum_{E \in \mathcal{E}} Pr(A|E) Pr_e(E) \quad (1)$$

A probability function Pr^* on the probability space $\langle \Omega, 2^\Omega \rangle$ is said to be obtained from Pr by *the principle of probability kinematics* on $\mathcal{E}$ if, for any $E \in \mathcal{E}$,

$$Pr^*(A|E) = Pr(A|E) \text{ for every event } A \subseteq \Omega.$$

In other words, the principle of probability kinematics assumes that the conditional probability in every event A given any $E \in \mathcal{E}$ remains unchanged. This concept was proposed by Jeffrey [JEFFREY, 1983] to capture the notion that, even though Pr^* and Pr disagree on the probabilities of events in the *coarser* Boolean algebra $\mathcal{B}$, they agree on their *relevance* to every event A in $\mathcal{A}$.

Actually the above posterior probability function Pr' proposed by Jeffrey's rule in Eq. (1) is the *unique* probability revision Pr^* which satisfies the following two requirements [CHAN AND DARWICHE, 2003]:

- (**C1**): probability kinematics on $\mathcal{E}$: for any $E \in \mathcal{B}$, $Pr^*(A|E) = Pr(A|E)$ for all $A \subseteq \Omega$;
- (**C2**): $Pr^*(E) = Pr_e(E)$ for all $E \in \mathcal{B}$.

A distance measure D can be defined for probability functions as follows [CHAN AND DARWICHE, 2002]: for any two probability functions Pr_1 and Pr_2,

$$D(Pr_1, Pr_2) = \ln max_{\omega \in \Omega} \frac{Pr_2(\omega)}{Pr_1(\omega)} - \ln min_{\omega \in \Omega} \frac{Pr_2(\omega)}{Pr_1(\omega)}$$

Among all the probability functions that satisfy the above requirement (**C2**), the posterior probability function Pr' proposed by Jeffrey's rule is the *closest* to Pr according to this distance measure [CHAN AND DARWICHE, 2003].

3 JEFFREY'S RULE IN DEMPSTER-SHAFER THEORY

3.1 BELIEF FUNCTIONS

Let Ω be a frame of discernment and $\mathcal{A} = 2^\Omega$ be the Boolean algebra of events. A *belief function* is a function $bel : \mathcal{A} \to [0, 1]$ satisfying the following conditions:

1. $bel(\emptyset) = 0$;
2. $bel(\Omega) = 1$; and
3. $bel(\bigcup_{i=1}^n A_i) = \sum_{\emptyset \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} bel(\cap_{i \in I} A_i)$ where $A_i \in \mathcal{A}$ for all $i \in \{1, \cdots, n\}$.

A *mass assignment* (or *mass function*) is a mapping $m : \mathcal{A} \to [0, 1]$ satisfying

$$m(\emptyset) = 0, \sum_{A \in \mathcal{A}} m(A) = 1.$$

Shafer [SHAFER, 1976] has shown that a mapping $f : \mathcal{A} \to [0, 1]$ is a belief function if and only if its Möbius transform is a mass assignment. In other words, if $m : \mathcal{A} \to [0, 1]$ is a mass assignment, then it determines a belief function $bel : \mathcal{A} \to [0, 1]$ as follows:

$$bel(A) = \sum_{B \subseteq A} m(B) \text{ for all } A \in \mathcal{A}.$$

Moreover, given a belief function bel, we can obtain its corresponding mass function m as follows:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} bel(B), \text{ for all } A \in \mathcal{A}.$$

Intuitively, for a subset event A, $m(A)$ measures the belief that an agent commits *exactly* to A, not the total belief that an agent commits to A.

3.2 JEFFREY'S RULE IN THE THEORY OF EVIDENCE

In order to introduce the principle of belief kinematics, we need to set up a setting in terms of *refinements and coarsenings* of frames of discernments. The idea that one frame Ω of discernment is obtained from another frame Θ of discernment by splitting some or all of the elements of Θ may be represented mathematically by specifying, for each $\theta \in \Theta$, the subset $\omega(\{\theta\})$ of Ω consisting of those possibilities into which θ has been split. For this representation to be sensible, we need only require that the sets $\omega(\{\theta\})$ should constitute a disjoint partition of Ω. Given such a disjoint partition $\omega(\{\theta\})$, we may set

$$\omega(A) = \bigcup_{\theta \in A} \omega(\{\theta\})$$

for each $A \subseteq \Theta$; $\omega(A)$ will consist of all the possibilities in Ω that are obtained by splitting the elements of A, and the mapping $\omega : 2^{\Theta} \to 2^{\Omega}$ that is thus defined will provide a thorough description of the splitting. Such a mapping ω is called a *refining*. Whenever $\omega : 2^{\Theta} \to 2^{\Omega}$ is a refining, we call Ω a *refinement* of Θ and Θ a *coarsening* of Ω.

In this paper, we are particularly interested in the case when Θ is the set of equivalence classes with respect to some partition Π of Ω. So the mapping $\omega(\{\Pi(w)\}) = \Pi(w)$ for each $w \in \Omega$ is a refining and Θ is a coarsening of Ω where $\Pi(w)$ is the equivalence class of w. We denote this special coarsening Θ of Ω as $\Omega/_{\Pi}$. On the other hand, Ω/Π may be regarded as a subalgebra $\mathcal{B}$ of the powerset of Ω with the set of atoms of $\mathcal{B}$ forming the partition Π of Ω. Our following definition of Jeffrey's rules in Dempster-Shafer theory is in terms of this type of presentation of the coarsening Ω/Π as $\langle \Omega, \mathcal{B} \rangle$. For example, $\Pi = \{\{w_1, w_2\}, \{w_3, w_4\}, \{w_5, w_6\}\}$ is a partition of $\Omega = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. Then the associated subalgebra $\mathcal{B}$ consists of the sets $\bigcup_{B \subseteq \Pi} B$ with the atoms $\{w_1, w_2\}, \{w_3, w_4\}$ and $\{w_5, w_6\}$ in $\mathcal{B}$.

For each $A \subseteq \Omega$, we define

$$\mathbf{B}(A) := \bigcap \{B \in \mathcal{B} : A \subseteq B\}$$

In other words, $\mathbf{B}(A)$ is the least element of $\mathcal{B}$ that contains A as a subset and hence is called the *upper approximation* of A in $\mathcal{B}$ [SMETS, 1993A]. For the above example, if $A = \{w_1, w_3, w_5\}$, then $\mathbf{B}(A) = \Omega$.

Let $\langle \Omega, \mathcal{B} \rangle$ be a coarsening of Ω where $\mathcal{B}$ is a subalgebra of the powerset 2^{Ω} of Ω with its atoms forming a partition of Ω. Suppose that $bel : 2^{\Omega} \to [0, 1]$ is a belief function on Ω with m as its corresponding mass assignment. Then *the derived mass assignment* $m^{in}_{\mathcal{B}}$ *on the coarsening* $\langle \Omega, \mathcal{B} \rangle$ can be obtained through the following formula: for any $B \in \mathcal{B}$,

$$m^{in}_{\mathcal{B}}(B) = \sum_{\mathbf{B}(A)=B, A \subseteq \Omega} m(A)$$

It is easy to see that, in the coarsening frame $\langle \Omega, \mathcal{B} \rangle$, $m^{in}_{\mathcal{B}}(B)$ measures the belief that commits exactly to B, not to any subset of B in $\mathcal{B}$. Let $bel^{in}_{\mathcal{B}}$ denote the corresponding belief function. It is easy to check that, for any $B \in \mathcal{B}$, $bel^{in}_{\mathcal{B}}(B) = bel(B)$. Intuitively, $bel^{in}_{\mathcal{B}}$ is the derived belief function on the coarsening frame of discernment with less distinctions. The beliefs in the same propositions in these two different frames with different distinctions should be the same as each other [SMETS, 1993B]. Correspondingly, since the resolution degree of the attention of the coarsening frame decreases, the mass assignment m has to change into $m^{in}_{\mathcal{B}}$.

For any A and B such that $A \subseteq B \in \mathcal{B}$ and $A \subseteq \Omega$, let $m_{/B}(A)$ denote $\sum_{E \subseteq \overline{B}} m(A \cup E)$ and

$$m^{out}_{\mathcal{B}}(B) = \sum_{\mathbf{B}(E)=B, E \subseteq \Omega} m_{/B}(E).$$

It is easy to see that, if B is an atom of the subalgebra $\mathcal{B}$, then $m^{in}_{\mathcal{B}}(B) = bel(B)$ and $m^{out}_{\mathcal{B}}(B) = pl(B)$. Now we define two different *conditional belief functions* $bel^{in}_{\mathcal{B}}(\cdot|B)$ and $bel^{out}_{\mathcal{B}}(\cdot|B)$ on a given $B \in \mathcal{B}$ according to the above two different definitions of mass functions $m^{in}_{\mathcal{B}}$ and $m^{out}_{\mathcal{B}}$ on $\mathcal{B}$, respectively: for any $A \subseteq \Omega$,

(1) (Inner conditioning)

$$bel^{in}_{\mathcal{B}}(A|B) := \begin{cases} \frac{\sum_{A' \subseteq A, \mathbf{B}(A')=B} m(A')}{m^{in}_{\mathcal{B}}(B)}, \text{if } m^{in}_{\mathcal{B}}(B) \neq 0, \\ \frac{|\{A' \subseteq A : \mathbf{B}(A')=B\}|}{|\{A' \subseteq \Omega : \mathbf{B}(A')=B\}|}, \text{if } m^{in}_{\mathcal{B}}(B) = 0 \end{cases}$$

(2) (Outer conditioning)

$$bel^{out}_{\mathcal{B}}(A|B) := \begin{cases} \frac{\sum_{A' \subseteq A, \mathbf{B}(A')=B} m_{/B}(A')}{m^{out}_{\mathcal{B}}(B)}, \\ \quad \text{if } m^{out}_{\mathcal{B}}(B) \neq 0, \\ \frac{|\{A' \subseteq A : \mathbf{B}(A' \cap B)=B\}|}{|\{A' \subseteq \Omega : \mathbf{B}(A' \cap B)=B\}|}, \\ \quad \text{if } m^{out}_{\mathcal{B}}(B) = 0 \end{cases}$$

Note that both $bel^{in}_{\mathcal{B}}(\cdot|B)$ and $bel^{out}_{\mathcal{B}}(\cdot|B)$ are belief functions on 2^{Ω} for any $B \in \mathcal{B}$.

The superscripts ^{in} and ^{out} in the above notations are designated for the following two proposed revision rules: inner revision and outer revision. In particular, when B is an atom in the algebra $\mathcal{B}$, the above defined $bel^{in}_{\mathcal{B}}(\cdot|B)$ and $bel^{out}_{\mathcal{B}}(\cdot|B)$, are essentially the geometric conditionalization and the Dempster conditionalization of A on B, respectively. However, the essential difference of our above definitions of conditional belief functions from the well-known Dempster's rule of conditioning $bel(\cdot|B)$ [SHAFER, 1976] is that they depend on the coarsening frame and hence on the degree of resolution of the attention while Dempster's rule of conditioning does not and is derived from Dempster's rule of combination. Moreover, Dempster's rule of combination relies on a basic assumption that the combined evidences (or beliefs) play the same role and hence the combination operation is *symmetric*. In contrast, in our study of revision of beliefs using uncertain evidence, we treat uncertain evidence in terms of *effect* it has on beliefs once accepted, which is a function of both evidence strength and beliefs held before the evidence is obtained. Hence the prior beliefs and uncertain evidence are intrinsically asymmetric. In this sense, our definition is a natural generalization of the classical Bayesian definition of conditional probabilities.

Lemma 3.1 *Let bel be a belief function on Ω with m its corresponding mass assignment and $\langle \Omega, \mathcal{B} \rangle$ a coarsening as above.*

1. *For any $A \subseteq \Omega$,*

$$bel(A) = \sum_{B \in \mathcal{B}} bel^{in}_{\mathcal{B}}(A|B) m^{in}_{\mathcal{B}}(B);$$

2. *If m_e is a mass assignment on $\langle \Omega, \mathcal{B} \rangle$, then the function $bel' : 2^{\Omega} \to [0,1]$ defined as follows, for any $A \subseteq \Omega$,*

$$bel'(A) = \sum_{B \in \mathcal{B}} bel^{in}_{\mathcal{B}}(A|B) m_e(B)$$

is a belief function. In particular, for each $B \in \mathcal{B}$,

$$bel'(B) = \sum_{B' \in \mathcal{B}, B' \subseteq B} m_e(B').$$

In other words, if bel_e is the corresponding belief function of m_e on $\langle \Omega, \mathcal{B} \rangle$, then $bel'(B) = bel_e(B)$ for all $B \in \mathcal{B}$; namely, m_e is exactly the derived *mass assignment $(m')^{in}_{\mathcal{B}}$ of bel' on the coarsening frame $\langle \Omega, \mathcal{B} \rangle$.*

Proof. The first part is obvious. And the second part follows from the following fact: for any $B, B' \in \mathcal{B}$,

$$bel^{in}_{\mathcal{B}}(B'|B) := \begin{cases} 1, & \text{if } B \subseteq B' \\ 0 & \text{otherwise.} \end{cases}$$

QED

Lemma 3.2 *Let bel be a belief function on Ω with m its corresponding mass assignment and $\langle \Omega, \mathcal{B} \rangle$ a coarsening as above. If m_e is a mass assignment on $\langle \Omega, \mathcal{B} \rangle$, then the function $bel' : 2^{\Omega} \to [0,1]$ defined as follows, for any $A \subseteq \Omega$,*

$$bel'(A) = \sum_{B \in \mathcal{B}} bel^{out}_{\mathcal{B}}(A|B) m_e(B)$$

is a belief function. In particular, for each $B \in \mathcal{B}$,

$$bel'(B) = \sum_{B' \in \mathcal{B}, B' \subseteq B} m_e(B').$$

In other words, if bel_e is the corresponding belief function of m_e on $\langle \Omega, \mathcal{B} \rangle$, then $bel'(B) = bel_e(B)$ for all $B \in \mathcal{B}$; namely, m_e is exactly the derived *mass assignment $(m')^{in}_{\mathcal{B}}$ of bel' on the coarsening frame $\langle \Omega, \mathcal{B} \rangle$.*

Proof. The first part is clear and the second follows from the following fact: for any $B, B' \in \mathcal{B}$,

$$bel^{out}_{\mathcal{B}}(B'|B) := \begin{cases} 1, & \text{if } B \subseteq B' \\ 0 & \text{otherwise.} \end{cases}$$

QED

Consider the problem of revising the belief function bel given uncertain evidence relating to a coarsening of Ω, which is represented as $\langle \Omega, \mathcal{B} \rangle$. One method of specifying the uncertain evidence is through the *effect* that it would have on beliefs once accepted. Specifically, according to the method, we have to specify uncertain evidence by providing the following constraint:

$$m'(B) = q_B, \text{ for each } B \in \mathcal{B} \tag{2}$$

where m' denotes the corresponding mass assignment of the new belief function bel' that results from accepting the given evidence. Also the specification can be represented as another belief function bel_e on $\langle \Omega, \mathcal{B} \rangle$ with m_e its corresponding mass assignment such that $m_e(B) = q_B$ for all $B \in \mathcal{B}$. To revise the belief function bel, we must therefore choose a *unique posterior* belief function that satisfies the above constraint. In order to achieve the uniqueness, we define next two forms of *belief kinematics*, the evidence-theoretic counterpart of the well-known probability kinematics [JEFFREY, 1983].

Definition 3.3 Suppose that bel and bel' are two belief functions on Ω, and m and m' are their corresponding mass assignments. Let $\langle \Omega, \mathcal{B} \rangle$ be a coarsening of Ω. The belief function bel' is said to be obtained from bel by *inner belief kinematics* on $\langle \Omega, \mathcal{B} \rangle$ if, for any $B \in \mathcal{B}$,

$$(bel')^{in}_{\mathcal{B}}(A|B) = bel^{in}_{\mathcal{B}}(A|B) \text{ for all } A \subseteq \Omega; \tag{3}$$

and it is said to be obtained from bel by *outer belief kinematics* on $\langle \Omega, \mathcal{B} \rangle$ if, for any $B \in \mathcal{B}$,

$$(bel')^{in}_{\mathcal{B}}(A|B) = bel^{out}_{\mathcal{B}}(A|B) \text{ for all } A \subseteq \Omega. \tag{4}$$

$\triangleleft$

Intuitively, the above principle of belief kinematics on $\langle \Omega, \mathcal{B} \rangle$ says that, even though bel and bel' may disagree on propositions on $\langle \Omega, \mathcal{B} \rangle$, they agree on their relevance to every event $A \subseteq \Omega$.

Now we define two revisions proposed by *Jeffrey's rule* as follows: for any $A \subseteq \Omega$,

1. $bel'(A) = \sum_{B \in \mathcal{B}} bel^{in}_{\mathcal{B}}(A|B) q_B$; and
2. $bel'(A) = \sum_{B \in \mathcal{B}} bel^{out}_{\mathcal{B}}(A|B) q_B$

According to Lemmas 3.1 and 3.2, both revisions satisfy the above constraint (2), and are indeed belief functions. These two revisions are called *inner and outer revisions* and the resulting belief functions are denoted as $bel^{in'}$ and $bel^{out'}$ by adding the corresponding superscripts ^{in} and ^{out}, respectively. It is easy to see that the well-known Jeffrey's rule for probability functions is a special case of our more general rules here for belief functions. So, our Jeffrey's rules satisfy Constraint 1.

Theorem 3.4 *The new belief function $bel^{in'}$ given above is the one and only belief function that satisfies the constraint in Eq. (2) and that is obtained from bel by inner belief kinematics on the coarsening frame $\langle \Omega, \mathcal{B} \rangle$.*

Proof. According to Lemma 3.1, it suffices to show that the new posterior belief function $bel^{in'}$ obtained through Jeffrey's rule satisfies the condition for inner belief kinematics: for any $A \subseteq \Omega, B \in \mathcal{B}$,

$$bel^{in}_{\mathcal{B}}(A|B) = (bel^{in'})^{in}_{\mathcal{B}}(A|B)$$

First note that, for any $A \subseteq \Omega$,

$$m^{in'}(A) = \textstyle\sum_{B \in \mathcal{B}} \frac{q_B}{m^{in}_{\mathcal{B}}(B)} l^{in}_B(A)$$

where $m^{in'}$ is the corresponding mass function of $bel^{in'}$ and

$$l^{in}_B(A) := \begin{cases} m(A), & \text{if } \mathbf{B}(A) = B \\ 0 & \text{otherwise.} \end{cases}$$

This follows directly from the following reasoning:

$$\begin{aligned} \sum_{A' \subseteq A} m^{in'}(A') &= \sum_{A' \subseteq A} (\sum_{B \in \mathcal{B}} \frac{q_B}{m^{in}_{\mathcal{B}}(B)} l^{in}_B(A')) \\ &= \sum_{B \in \mathcal{B}} (\sum_{A' \subseteq A} \frac{q_B}{m^{in}_{\mathcal{B}}(B)} l^{in}_B(A')) \\ &= \sum_{B \in \mathcal{B}} (\frac{\sum_{A' \subseteq A} l^{in}_B(A')}{m^{in}_{\mathcal{B}}(B)} q_B) \\ &= \sum_{B \in \mathcal{B}} (\frac{\sum_{A' \subseteq A, \mathbf{B}(A') = B} m(A')}{m^{in}_{\mathcal{B}}(B)} q_B) \\ &= \sum_{B \in \mathcal{B}} bel^{in}_{\mathcal{B}}(A|B) q_B \\ &= bel^{in'}(A) \end{aligned}$$

According to Lemma 3.1, $(m^{in'})^{in}_{\mathcal{B}}(B) = \sum_{\mathbf{B}(A)=B, A \subseteq \Omega} m^{in'}(A) = q_B$ where $(m^{in'})^{in}_{\mathcal{B}}$ is the derived mass assignment of $m^{in'}$ on $\langle \Omega, \mathcal{B} \rangle$. Next we use this expression of $m^{in'}$ to proceed as follows:

$$\begin{aligned} (bel^{in'})^{in}_{\mathcal{B}}(A|B) &= \frac{1}{q_B} \sum_{A' \subseteq A, \mathbf{B}(A') = B} m^{in'}(A') \\ &= \frac{1}{q_B} \sum_{A' \subseteq A, \mathbf{B}(A') = B} \sum_{B' \in \mathcal{B}} \frac{q_{B'} l^{in}_{B'}(A')}{m^{in}_{\mathcal{B}}(B')} \\ &= \frac{1}{q_B} \sum_{A' \subseteq A, \mathbf{B}(A') = B} \frac{q_B}{m^{in}_{\mathcal{B}}(B)} m(A') \\ &= \frac{\sum_{A' \subseteq A, \mathbf{B}(A') = B} m(A')}{m^{in}_{\mathcal{B}}(B)} \\ &= bel^{in}_{\mathcal{B}}(A|B) \end{aligned}$$

QED

Theorem 3.5 *The new belief function $bel^{out'}$ given above is the one and only belief function that satisfies the constraint in Eq. (2) and that is obtained from bel by outer belief kinematics on the coarsening frame $\langle \Omega, \mathcal{B} \rangle$.*

Proof. According to Part (1) of Lemma 3.1 and Lemma 3.2, it suffices to show that the new posterior belief function obtained through Jeffrey's rule satisfies the condition in outer belief kinematics, i.e., for any $B \in \mathcal{B}$,

$$(bel^{out'})^{in}_{\mathcal{B}}(A|B) = bel^{out}_{\mathcal{B}}(A|B) \text{ for all } A \subseteq \Omega.$$

But this follows from a similar argument to that in the proof of Theorem 3.4.

QED

The above propositions tell us that the two Jeffrey's rules are obtained from belief kinematics and hence satisfies Constraint 3.

Example 3.6 The following example is adapted from the original one by Jeffrey [JEFFREY, 1983] (also [CHAN AND DARWICHE, 2003]). Assume that we are given a piece of cloth, where its color can be one of: green, blue, or violet. We want to know whether, on the next day, the cloth will be sold, or not sold. We denote the possible states as follows:

$$\begin{aligned} w_{1,g} &= (sold, green), & w_{0,g} &= (\text{not sold}, green) \\ w_{1,b} &= (sold, blue), & w_{0,b} &= (\text{not sold}, blue) \\ w_{1,v} &= (sold, violet), & w_{0,v} &= (\text{not sold}, violet) \end{aligned}$$

Our original belief bel is given by the following mass assignment m on $\Omega := \{w_{n,c} : n \in \{0,1\}, c \in \{b, g, v\}\}$:

$$\begin{aligned} & m(\{w_{1,g}\}) = m(\{w_{1,b}\}) = m(\{w_{1,v}\}) = 0.1 \\ & m(\{w_{0,g}\}) = m(\{w_{0,b}\}) = m(\{w_{0,v}\}) = 0.15 \\ & m(\{w_{1,g}, w_{0,b}\}) = m(\{w_{1,b}, w_{0,v}\}) = 0.1 \\ & m(\{w_{1,g}, w_{1,v}\}) = 0.05 \end{aligned}$$

The possible states w_c of colors denote $\{w_{1,c}, w_{0,c}\}$ for all $c \in \{g, v, b\}$. Let $\mathcal{B}$ be a subalgebra of the powerset of Ω that consists of the propositions of the form $\bigcup_{B\subseteq\{w_b,w_g,w_v\}} B$. It is easy to see that $\langle\Omega, \mathcal{B}\rangle$ is a coarsening of $\langle\Omega, 2^\Omega\rangle$.

The derived mass assignment $m^{in}_{\mathcal{B}}$ on the coarsening frame $\langle\Omega, \mathcal{B}\rangle$ can be computed as follows:

$$\begin{aligned} &m^{in}_{\mathcal{B}}(\{w_g\}) = m^{in}_{\mathcal{B}}(\{w_b\}) = m^{in}_{\mathcal{B}}(\{w_v\}) = 0.25 \\ &m^{in}_{\mathcal{B}}(\{w_g, w_b\}) = m^{in}_{\mathcal{B}}(\{w_b, w_v\}) = 0.1 \\ &m^{in}_{\mathcal{B}}(\{w_g, w_v\}) = 0.05, \qquad m^{in}_{\mathcal{B}}(\{w_b, w_g, w_v\}) = 0 \end{aligned}$$

Now we consider the conditional beliefs of a given proposition $A := \{w_{1,g}, w_{0,b}, w_{1,v}\}$ as an illustration. We obtain $bel(A) = 0.5$. According to our previous definition of inner conditioning, we have

$$\begin{aligned} bel^{in}_{\mathcal{B}}(A|\{w_g\}) &= \frac{2}{5}, \qquad bel^{in}_{\mathcal{B}}(A|\{w_b\}) = \frac{3}{5} \\ bel^{in}_{\mathcal{B}}(A|\{w_v\}) &= \frac{2}{5}, \qquad bel^{in}_{\mathcal{B}}(A|\{w_g, w_b\}) = 1 \\ bel^{in}_{\mathcal{B}}(A|\{w_g, w_v\}) &= 1, \qquad bel^{in}_{\mathcal{B}}(A|\{w_b, w_v\}) = 0 \\ bel^{in}_{\mathcal{B}}(A|\{w_b, w_v, w_g\}) &= \frac{1}{27} \end{aligned}$$

Assume that we now inspect the cloth by candlelight, and conclude that our belief on the color of the cloth should be:

$$\begin{aligned} &bel_e(\{w_g\}) = bel_e(\{w_b\}) = bel_e(\{w_v\}) = 0.2 \\ &bel_e(\{w_g, w_b\}) = bel_e(\{w_b, w_v\}) = 0.5 \\ &bel_e(\{w_g, w_v\}) = 0.6 \end{aligned}$$

The corresponding mass assignment m_e is as follows:

$$\begin{aligned} &m_e(\{w_g\}) = m_e(\{w_b\}) = m_e(\{w_v\}) = 0.2 \\ &m_e(\{w_g, w_b\}) = m_e(\{w_b, w_v\}) = 0.1 \\ &m_e(\{w_g, w_v\}) = 0.2 \end{aligned}$$

So, according to our definition of Jeffrey's rule, we have the new inner revision of belief in the event A:

$$bel^{in'}(A) = \textstyle\sum_{B\in\mathcal{B}} bel^{in}_{\mathcal{B}}(A|B) m_e(B) = 0.58$$

Now we compute the outer revision of the belief in A. The mass assignment $m^{out}_{\mathcal{B}}$ on the coarsening frame $\langle\Omega, \mathcal{B}\rangle$ can be computed as follows:

$$\begin{aligned} &m^{out}_{\mathcal{B}}(\{w_g\}) = 0.4 = m^{out}_{\mathcal{B}}(\{w_v\}), m^{out}_{\mathcal{B}}(\{w_b\}) = 0.45 \\ &m^{out}_{\mathcal{B}}(\{w_g, w_b\}) = 0.1 = m^{out}_{\mathcal{B}}(\{w_b, w_v\}) \\ &m^{out}_{\mathcal{B}}(\{w_g, w_v\}) = 0.05, \qquad m^{out}_{\mathcal{B}}(\{w_b, w_g, w_v\}) = 0 \end{aligned}$$

According to our previous definition of outer conditioning, we have

$$\begin{aligned} bel^{out}_{\mathcal{B}}(A|\{w_g\}) &= \frac{1}{4}, bel^{out}_{\mathcal{B}}(A|\{w_b\}) = \frac{2}{9} \\ bel^{out}_{\mathcal{B}}(A|\{w_v\}) &= \frac{1}{8}, bel^{out}_{\mathcal{B}}(A|\{w_g, w_b\}) = 1 \\ bel^{out}_{\mathcal{B}}(A|\{w_g, w_v\}) &= 1, bel^{out}_{\mathcal{B}}(A|\{w_b, w_v\}) = 0 \\ bel^{out}_{\mathcal{B}}(A|\{w_b, w_v, w_g\}) &= \frac{1}{27} \end{aligned}$$

Hence we have

$$bel^{out'}(A) = \textstyle\sum_{B\in\mathcal{B}} bel^{out}_{\mathcal{B}}(A|B) m_e(B) = \frac{151}{360}$$

4 MEASURES FOR BOUNDING BELIEF CHANGES

One important question relating to belief revision is that of measuring the extent to which a revision disturbs existing beliefs. In the following, we simulate the work by Chan and Darwiche [CHAN AND DARWICHE, 2002] by proposing for each Jeffrey's revision rule a distance measure for belief functions which can be used to bound the amount of belief changes induced by this revision using uncertain evidence and show that, according to this measure, the posterior belief function obtained by the corresponding belief kinematics is the closest to the original one among all belief functions that satisfy the constraint in Eq. (2).

Definition 4.1 Let bel and bel' be two belief functions over the same frame Ω of discernment. We define a measure between bel and bel' as follows:

$$D^{in}(bel, bel') = \ln max_{A\subseteq\Omega} \frac{m'(A)}{m(A)} - \ln min_{A\subseteq\Omega} \frac{m'(A)}{m(A)}$$

where $\frac{0}{0}$ is defined to be 1. It is easy to check that D^{in} is a distance (or metric), satisfying the three properties of distance and, whenever there is a subset A for which $m(A) = 0$ and $m'(A) > 0$ or vice versa, the distance $D^{in}(bel, bel')$ for the corresponding belief functions is equal to infinity. $\triangleleft$

Lemma 4.2 *Let $\langle\Omega, \mathcal{B}\rangle$ be a coarsening of $\langle\Omega, 2^\Omega\rangle$. Assume that $bel^{in'}$ is obtained from bel by applying Jeffrey's rule according to inner belief kinematics (Eq. (3)), given the uncertain evidence specified by the set of posterior beliefs $(m^{in'})^{in}_{\mathcal{B}}(B) = q_B$, for $B \in \mathcal{B}$ where $(m^{in'})^{in}_{\mathcal{B}}$ is the derived mass assignment of $m^{in'}$, the corresponding mass assignment of $bel^{in'}$, on $\langle\Omega, \mathcal{B}\rangle$.*

1. *For any $A \subseteq \Omega$, if $\boldsymbol{B}(A) = B$, then*

$$\frac{m^{in'}(A)}{m(A)} = \frac{(m^{in'})^{in}_{\mathcal{B}}(B)}{m^{in}_{\mathcal{B}}(B)}.$$

2. *The distance $D^{in}(bel, bel')$ between bel and $bel^{in'}$ is given by*

$$D^{in}(bel, bel^{in'}) = \ln max_{B \in \mathcal{B}} \frac{q_B}{m^{in}_{\mathcal{B}}(B)} - \ln min_{B \in \mathcal{B}} \frac{q_B}{m^{in}_{\mathcal{B}}(B)}$$

Proof. The first part follows from the following observation: for any $A \subseteq \Omega$ and $B \in \mathcal{B}$,

$$\frac{\sum_{\mathbf{B}(A')=B, A' \subseteq A} m^{in'}(A')}{\sum_{\mathbf{B}(A')=B, A' \subseteq A} m(A')} = \frac{(bel^{in'})^{in}_{\mathcal{B}}(A|B)(m^{in'})^{in}_{\mathcal{B}}(B)}{bel^{in}_{\mathcal{B}}(A|B) m^{in}_{\mathcal{B}}(B)} = \frac{(m^{in'})^{in}_{\mathcal{B}}(B)}{m^{in}_{\mathcal{B}}(B)}$$

The second equality comes from the condition for inner belief kinematics. QED

The following theorem says that the principle of inner belief kinematics can be viewed as a principle for minimizing belief change with respect to the metric D^{in}.

Theorem 4.3 *For the belief functions bel and $bel^{in'}$ in Lemma 4.2, $bel^{in'}$ is the closest to bel according to the above distance measure D^{in} among all possible belief functions that agree with $bel^{in'}$ on the propositions in the subalgebra $\mathcal{B}$.*

Proof. Suppose that m is the corresponding mass assignment of bel. Let bel'' be any belief function with m'' as its corresponding mass assignment that satisfies the constraint: $(m'')^{in}_{\mathcal{B}}(B) = (m^{in'})^{in}_{\mathcal{B}}(B)$ for all $B \in \mathcal{B}$. Let $B_{max} = argmax_{B \in \mathcal{B}}(\frac{(m^{in'})^{in}_{\mathcal{B}}(B)}{m^{in}_{\mathcal{B}}(B)})$ and $B_{min} = argmin_{B \in \mathcal{B}}(\frac{(m^{in'})^{in}_{\mathcal{B}}(B)}{m^{in}_{\mathcal{B}}(B)})$. Define $r_{max} = \max_{A \subseteq \Omega} \frac{m''(A)}{m(A)}$. Then we have the following inequality:

$$\begin{aligned} r_{max} m^{in}_{\mathcal{B}}(B_{max}) &= r_{max} \sum_{\mathbf{B}(A)=B_{max}, A \subseteq \Omega} m(A) \\ &\geq \sum_{\mathbf{B}(A)=B_{max}, A \subseteq \Omega} \frac{m''(A)}{m(A)} m(A) \\ &= \sum_{\mathbf{B}(A)=B_{max}, A \subseteq \Omega} m''(A) \\ &= (m'')^{in}_{\mathcal{B}}(B_{\max}) \\ &= (m^{in'})^{in}_{\mathcal{B}}(B_{\max}) \end{aligned}$$

So we have shown that $r_{max} \geq \frac{(m^{in'})^{in}_{\mathcal{B}}(B_{max})}{m^{in}_{\mathcal{B}}(B_{max})}$. Similarly, we can define $r_{min} = \min_{A \subseteq \Omega} \frac{m''(A)}{m(A)}$ and show that $r_{min} \leq \frac{(m^{in'})^{in}_{\mathcal{B}}(B_{min})}{m^{in}_{\mathcal{B}}(B_{min})}$. Therefore, the distance measure between bel and bel'' is:

$$\begin{aligned} D^{in}(bel, bel'') &= \ln r_{max} - \ln r_{min} \\ &\geq \ln \frac{(m^{in'})^{in}_{\mathcal{B}}(B_{max})}{m^{in}_{\mathcal{B}}(B_{max})} - \ln \frac{(m^{in'})^{in}_{\mathcal{B}}(B_{min})}{m^{in}_{\mathcal{B}}(B_{min})} \\ &= \ln max_{B \in \mathcal{B}} \frac{(m^{in'})^{in}_{\mathcal{B}} B}{m^{in}_{\mathcal{B}}(B)} - \ln min_{B \in \mathcal{B}} \frac{(m^{in'})^{in}_{\mathcal{B}} B}{m^{in}_{\mathcal{B}}(B)} \\ &= D^{in}(bel, bel') \end{aligned}$$

The last equality follows from Lemma 4.2.

QED

Now we define a distance D^{out} for the outer revision. The essential difference of D^{out} from the above D^{in} for the inner revision is that D^{out} depends on the associated coarsening frame.

Definition 4.4 Let bel and bel' be two belief functions over the same frame Ω of discernment. We define a measure between bel and bel' *with respect to a coarsening* $\langle \Omega, \mathcal{B} \rangle$ as follows:

$$D^{out}(bel, bel') = \ln max_{A \subseteq \Omega} \frac{m'(A)}{m_{/\mathbf{B}(A)}(A)} - \ln min_{A \subseteq \Omega} \frac{m'(A)}{m_{/\mathbf{B}(A)}(A)}$$

where $\frac{0}{0}$ is defined to be 1. It is easy to check that D^{out} is a distance (or metric), satisfying the three properties of distance. $\lhd$

Lemma 4.5 *Let $\langle \Omega, \mathcal{B} \rangle$ be a coarsening of $\langle \Omega, 2^{\Omega} \rangle$. Assume that $bel^{out'}$ is obtained from bel by applying Jeffrey's rule according to the outer belief kinematics, given the uncertain evidence specified by the set of posterior beliefs $(m^{out'})^{in}_{\mathcal{B}}(B) = q_B$, for $B \in \mathcal{B}$ where $(m^{out'})^{in}_{\mathcal{B}}$ is the derived mass assignment of $m^{out'}$, the corresponding mass assignment of $bel^{out'}$, on $\langle \Omega, \mathcal{B} \rangle$.*

1. *For any $A \subseteq \Omega$, if $\mathbf{B}(A) = B$, then*

$$\frac{m^{out'}(A)}{m_{/\mathbf{B}(A)}(A)} = \frac{(m^{out'})^{in}_{\mathcal{B}}(B)}{m^{out}_{\mathcal{B}}(B)}.$$

2. *The distance $D^{out}(bel, bel')$ between bel and bel' is given by*

$$D^{out}(bel, bel') = \ln max_{B \in \mathcal{B}} \frac{q_B}{m^{out}_{\mathcal{B}}(B)} - \ln min_{B \in \mathcal{B}} \frac{q_B}{m^{out}_{\mathcal{B}}(B)}$$

Proof. The first part follows from the following observation: for any $A \subseteq \Omega$ and $B \in \mathcal{B}$,

$$\frac{\sum_{\mathbf{B}(A')=B, A' \subseteq A} m^{out'}(A')}{\sum_{\mathbf{B}(A')=B, A' \subseteq A} m_{/\mathbf{B}(A')}(A')} = \frac{(bel^{out'})^{in}_{\mathcal{B}}(A|B)(m^{out'})^{in}_{\mathcal{B}}(B)}{bel^{out}_{\mathcal{B}}(A|B) m^{out}_{\mathcal{B}}(B)} = \frac{(m^{out'})^{in}_{\mathcal{B}}(B)}{m^{out}_{\mathcal{B}}(B)}$$

The second equality comes from the condition for outer belief kinematics. QED

The following theorem says that the principle of outer belief kinematics can be viewed as a principle for minimizing belief change with respect to D^{out}.

Theorem 4.6 *For the belief functions bel and $bel^{out'}$ in Lemma 4.5, $bel^{out'}$ is the closest to bel according to the above distance measure D^{out} among all possible belief functions that agree with $bel^{out'}$ on the propositions in the subalgebra $\mathcal{B}$.*

Proof. Suppose that m is the corresponding mass assignments of bel. Let bel'' be any belief function with m'' as its corresponding mass assignment that satisfies the constraint: $(m'')^{in}_{\mathcal{B}}(B) = (m^{out'})^{in}_{\mathcal{B}}(B)$ for all $B \in \mathcal{B}$. Let $B_{max} = argmax_{B\in\mathcal{B}}(\frac{(m^{out'})^{in}_{\mathcal{B}}(B)}{m^{out}_{\mathcal{B}}(B)})$ and $B_{min} = argmin_{B\in\mathcal{B}}(\frac{(m^{out'})^{in}_{\mathcal{B}}(B)}{m^{out}_{\mathcal{B}}(B)})$. Define $r_{max} = \max_{A\subseteq\Omega} \frac{m''(A)}{m_{/\mathbf{B}(A)}(A)}$. Then we have the following inequality:

$$
\begin{aligned}
& r_{max} m^{out}_{\mathcal{B}}(B_{max}) = r_{max} \sum_{\mathbf{B}(A)=B_{max}, A\subseteq\Omega} m_{/\mathbf{B}(A)}(A) \\
& \geq \sum_{\mathbf{B}(A)=B_{max}, A\subseteq\Omega} \frac{m''(A)}{m_{/\mathbf{B}(A)}(A)} m_{/\mathbf{B}(A)}(A) \\
& = \sum_{\mathbf{B}(A)=B_{max}, A\subseteq\Omega} m''(A) \\
& = (m'')^{in}_{\mathcal{B}}(B_{\max}) \\
& = (m^{out'})^{in}_{\mathcal{B}}(B_{\max})
\end{aligned}
$$

So we have shown that $r_{max} \geq \frac{(m^{out'})^{in}_{\mathcal{B}}(B_{max})}{m^{out}_{\mathcal{B}}(B_{max})}$. Similarly, we can define $r_{min} = \min_{A\subseteq\Omega} \frac{m''(A)}{m_{/\mathbf{B}(A)}(A)}$ and show that $r_{min} \leq \frac{(m^{out'})^{in}_{\mathcal{B}}(B_{min})}{m^{out}_{\mathcal{B}}(B_{min})}$. Therefore, the distance measure between bel and bel'' is:

$$
\begin{aligned}
& D^{out}(bel, bel'') \\
& = \ln max_{A\subseteq\Omega} \frac{m''(A)}{m_{/\mathbf{B}(A)}(A)} - \ln min_{A\subseteq\Omega} \frac{m''(A)}{m_{/\mathbf{B}(A)}(A)} \\
& = \ln r_{max} - \ln r_{min} \\
& \geq \ln \frac{(m^{out'})^{in}_{\mathcal{B}}(B_{max})}{m^{out}_{\mathcal{B}}(B_{max})} - \ln \frac{(m^{out'})^{in}_{\mathcal{B}}(B_{min})}{m^{out}_{\mathcal{B}}(B_{min})} \\
& = \ln max_{B\in\mathcal{B}} \frac{(m^{out'})^{in}_{\mathcal{B}} B}{m^{out}_{\mathcal{B}}(B)} - \ln min_{B\in\mathcal{B}} \frac{(m^{out'})^{in}_{\mathcal{B}} B}{m^{out}_{\mathcal{B}}(B)} \\
& = D^{out}(bel, bel')
\end{aligned}
$$

The last equality follows from Lemma 4.5.

QED

Example 4.7 Now, by using Lemmas 4.2 and 4.5, we compute the distances between bel and bel' in Example 3.6:

$$
\begin{aligned}
D^{in}(bel, bel') &= \ln 1 - \ln \frac{1}{4} \\
&= 2 \ln 2 \\
D^{out}(bel, bel') &= \ln 4 - \ln \frac{4}{9} \\
&= 3 \ln 3
\end{aligned}
$$

5 RELATED WORKS AND CONCLUSION

Although belief revision in probability theory is fully studied and researchers have generally agreed on the standard form of Jeffrey's rule, the corresponding revision rule in evidence theory has seldom been adequately addressed and there is not yet any standard form of this rule that has been universally recognized. Although all the forms of Jeffrey's rule in the theory of evidence in the literature are generalizations of this rule in probability theory, none of them satisfies all of the three natural constraints proposed in this paper. Usually they satisfy some constraints but do not satisfy the others. In particular, none of these forms in the literature has considered the revision of beliefs using uncertain evidence from the perspective in this paper viewing Jeffrey's rule as a form of the evidence-theoretic counterpart of probability kinematics, which should be the essence of Jeffrey's rule in Dempster-Shafer theory [WAGNER, 1992]. Moreover, none of those Jeffrey's rules in DS-theory in the literature has provided any distance measures for bounding belief changes due to revision and our work is the first to achieve that. Our distance measures for belief functions are adapted from the one for revision of probabilistic beliefs using uncertain evidence as *virtual certain evidence* according to Pearl's method [PEARL, 1988] and hence different from those distances for belief functions in the literature [JOUSSELME AND MAUPIN, 2012].

Jeffrey's rules for belief functions in the literature are proposed from different perspectives. Shafer [SHAFER, 1981] has studied Jeffery's rule. He proposed that its generalization can be found in Dempster's rule of combination. His proposal doesn't fit with Constraint 2 and we agree with Smets [SMETS, 1993A] that Constraint 2 is more important in the spirit of Jeffrey's updating than Shafer's proposal. In addition, Wagner [WAGNER, 1992] studied Jeffrey's rule in evidence theory from the perspective of viewing belief functions as lower envelopes. So his perspective is quite different from our proposal in term of mass functions.

Dubois and Prade [DUBOIS AND PRADE, 1991, DUBOIS AND PRADE, 1993] investigated updating and revision rules in a variety of uncertainty models including belief functions. They proposed the following form of Jeffrey's rule in DS theory:

$bel'(A) = \sum_{B\in\mathcal{B}} \frac{bel(A\cup\bar{B})-bel(\bar{B})}{pl(B)} m_e(B)$. This form is one of several Jeffrey's rules studied by Ichihashi and Tanaka ([ICHIHASHI AND TANAKA, 1989]). Ma and others ([MA ET AL., 2010] and in more detail [MA ET AL., 2011]) proposed three different revision rules, namely the inner, outer and modified outer revisions. In particular, their modified outer revision generalizes Jeffrey's rule of updating in probability theory, Dempster's rule of conditioning and a form of AGM revision. Their rules work in a more general setting when the incoming input is a *general* mass function. They consider the *information content* associated with an epistemic state represented by some belief function rather than the *full specification* as in our paper. A belief function bel_1 is less informed than another one bel_2 if bel_2 is a specialization of bel_1. They formalize the success postulate as requiring that the posterior belief function bel' be a specialization of the prior one bel. According to their viewpoint, if bel_1 and bel_2 are both defined on the same algebra $\mathcal{A}$ and bel_1 is a specialization of bel_2, then they are considered to be consistent with each other. However, according to our idea, they are inconsistent if they are not the same. We take the readaptation as revision [SMETS, 1993A]. Halpern [HALPERN, 2005] provided another form of belief-function revision rule: $bel'(A) = \sum_{i=1}^{n} bel_e(B_i)bel(A|B_i)$ where $(B_i)_{i=1}^{n}$ is a family of mutually exclusive and exhaustive subsets of Ω. His generalization is in terms of belief functions instead of mass functions. So it is quite different from ours. Moreover, bel' in his revision rule is not necessarily a belief function unless bel_e is a probability function. But, according to our proposal, uncertain evidence should be specified by a belief function bel_e. None of the above mentioned forms of Jeffrey's rules in DS-theory satisfies Constraint 2.

Our proposed Jeffrey's rules actually *improve* the two rules called source-conditioning and data-conditioning by Smets [SMETS, 1993A] especially the source-conditioning rule there. The motivation for the rule is not well justified and Smets' constraints for this rule are not well-defined. In Smets' Constraint **C2F**, bel should satisfy the requirement that, for any $X, Y \subseteq \Omega$, if $\mathbf{B}(X) = \mathbf{B}(Y)$, $\frac{bel(X)}{bel(Y)} = \frac{bel'(X)}{bel'(Y)}$. But generally bel' does not satisfy this requirement. Consider the above Example 3.6 and the proposition A. Let $A' = \{w_{0,g}, w_{1,b}, w_{0,v}\}$. Obviously, $\mathbf{B}(A) = \mathbf{B}(A') = \Omega$. However, $\frac{bel(A)}{bel(A')} = 1 \neq \frac{29}{18} = \frac{bel'(A)}{bel'(A')}$. The two forms of belief kinematics in this paper correct and improve the two constraints **C2F** and **C3F** in [SMETS, 1993A] (Parts (1) of Lemmas 4.2 4.5), respectively. Benferhat and others [BENFERHAT ET AL., 2011] studied Jeffrey's rule in a *possibilistic* framework using the possibilistic counterparts of probability kinematics, which is similar to our approach in this paper. But, our theory for belief functions here covers their approach in the quantitative possibilistic setting.

The following are some other constraints for defining Jeffrey's rules in Dempster-Shafer theory [MA ET AL., 2011]:

- (Constraint 4) When the incoming information is certain, the proposed Jeffrey's rule should be the same as Dempster's rule of conditioning.
- (Constraint 5) The proposed rule should satisfy some natural form of minimal change principle.
- (Constraint 6) The revision rule should preserve the new evidence.

We summarize our contributions in this paper by listing in a table the above major proposed Jeffrey's rules and their satisfied constraints:

Table 1: Summary

Rule \ Constraint	1	2	3	4	5	6
Shafer's rule	√			√		√
Modified outer revision rule by Ma et al.	√			√	√	√
Halpern's rule	√			√		
Smets' rule of source-conditioning	√	√				√
Smets' rule of data-conditioning	√	√		√		√
Our rule of inner revision	√	√	√		√	√
Our rule of outer revision	√	√	√	√	√	√

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References

[BENFERHAT ET AL., 2011] BENFERHAT, S., TABIA, K., AND SEDKI, K. (2011). JEFFREY'S RULE OF CONDITIONING IN A POSSIBILISTIC FRAMEWORK - AN ANALYSIS OF THE EXISTENCE AND UNIQUENESS OF THE SOLUTION. *Ann. Math. Artif. Intell.*, 61(3):185–202.

[Chan and Darwiche, 2002] Chan, H. and Darwiche, A. (2002). A distance measure for bounding probabilistic belief change. In Dechter, R. and Sutton, R. S., editors, *AAAI/IAAI*, pages 539–545. AAAI Press / The MIT Press.

[Chan and Darwiche, 2003] Chan, H. and Darwiche, A. (2003). On the revision of probabilistic beliefs using uncertain evidence. In Gottlob, G. and Walsh, T., editors, *IJCAI*, pages 99–105. Morgan Kaufmann.

[Dubois and Prade, 1991] Dubois, D. and Prade, H. (1991). Updating with belief functions, ordinal conditional functions and possibility measures. In Bonissone, P., M., H., L., K., and J., L., editors, *UAI*, pages 311–330. Elsevier.

[Dubois and Prade, 1993] Dubois, D. and Prade, H. (1993). Belief revision and updates in numerical formalisms: An overview, with new results for the possibilistic framework. In Bajcsy, R., editor, *IJCAI*, pages 620–625. Morgan Kaufmann.

[Halpern, 2005] Halpern, J. (2005). *Reasoning about Uncertainty*. MIT Press.

[Ichihashi and Tanaka, 1989] Ichihashi, H. and Tanaka, H. (1989). Jeffrey-like rules of conditioning for the dempster-shafer theory of evidence. *Int. J. Approx. Reasoning*, 3(2):143–156.

[Jeffrey, 1983] Jeffrey, R. (1983). *The Logic of Decision*. University of Chicago Press, second edition.

[Jousselme and Maupin, 2012] Jousselme, A. and Maupin, P. (2012). Distances in evidence theory: Comprehensive survey and generalizations. *Int. J. Approx. Reasoning*, 53(2):118–145.

[Ma et al., 2010] Ma, J., Liu, W., Dubois, D., and Prade, H. (2010). Revision rules in the theory of evidence. In *ICTAI (1)*, pages 295–302. IEEE Computer Society.

[Ma et al., 2011] Ma, J., Liu, W., Dubois, D., and Prade, H. (2011). Bridging jeffrey's rule, agm revision and dempster conditioning in the theory of evidence. *International J. on AI Tools*, 20 (4):691–720.

[Pearl, 1988] Pearl, J. (1988). *Probabilistic Reasoning in Intelligent Systems - Networks of Plausible Inference*. Morgan Kaufmann series in representation and reasoning. Morgan Kaufmann.

[Shafer, 1976] Shafer, G. (1976). *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, N.J.

[Shafer, 1981] Shafer, G. (1981). Jeffrey's rule of conditioning. *Philosophy of Sciences*, 48:337–362.

[Smets, 1993a] Smets, P. (1993a). Jeffrey's rule of conditioning generalized to belief functions. In Heckerman, D. and Mamdani, E. H., editors, *UAI*, pages 500–505. Morgan Kaufmann.

[Smets, 1993b] Smets, P. (1993b). Quantifying beliefs by belief functions: An axiomatic justification. In Bajcsy, R., editor, *IJCAI*, pages 598–605. Morgan Kaufmann.

[Wagner, 1992] Wagner, C. G. (1992). Generalizing jeffrey conditionalization. In Dubois, D. and Wellman, M. P., editors, *UAI*, pages 331–335. Morgan Kaufmann.