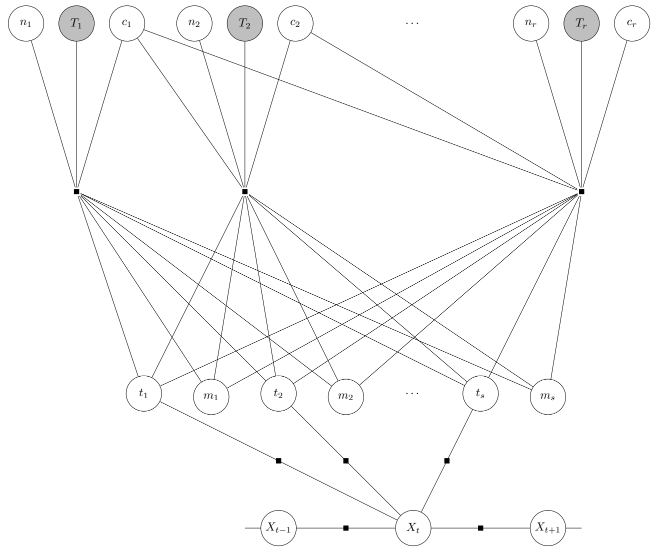


Figure 9: Factor graph corresponding to Figure 2

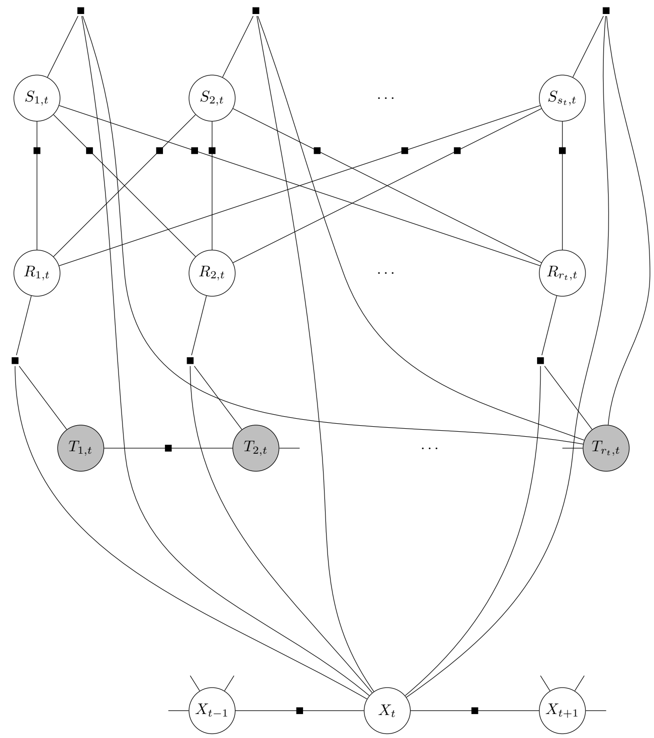


Figure 10: A factor graph representing the distribution of r earliest arriving signals in the bipartite model

A APPENDIX

A.1 MESSAGE PASSING EQUATIONS FOR BIPARTITE GRAPH

We use message passing for inference in the factor graph shown in Figure 10 where at time step t $T_1, T_2, \ldots, T_r$ are observed. In the following description of message passing at time step t we sometimes omit the t subscript for notational conveniences.

For each S_i, the message $\nu_{S_i \to S_i,R_j}$ to the factor $f(S_i, R_j)$ is

$$\nu_{S_i \to S_i,R_j} = \mu_{S_i,X_t,T_r \to S_i} \prod_{\substack{1 \le J \le r \\ J \ne j}} \mu_{S_i,R_J \to S_i} \tag{21}$$

the message $\mu_{S_i,R_j \to S_i}$ from the factor $f(S_i, R_j)$ is

$$\mu_{S_i,R_j \to S_i} = \sum_{R_j} f(S_i, R_j)\, \nu_{R_j \to S_i,R_j} \tag{22}$$

the message $\nu_{S_i \to S_i,X_t,T_r}$ to the factor $f(S_i, X_t, T_r)$ is

$$\nu_{S_i \to S_i,X_t,T_r} = \prod_{1 \le J \le r} \mu_{S_i,R_J \to S_i} \tag{23}$$

the message $\mu_{S_i,X_t,T_r \to S_i}$ from the factor $f(S_i, X_t, T_r)$ is

$$\mu_{S_i,X_t,T_r \to S_i} = \tag{24}$$

$$\int f(S_i, X_t, T_r)\, \nu_{X_t \to S_i,X_t,T_r} \nu_{T_r \to S_i,X_t,T_r}\, dX_t \tag{25}$$

For each R_j, the message $\nu_{R_j \to S_i,R_j}$ to the factor $f(S_i, R_j)$ is

$$\nu_{R_j \to S_i,R_j} = \tag{26}$$

$$\mu_{R_1,R_2,\ldots,R_r,N \to R_j} \mu_{T_j,R_j,X_t \to R_j} \prod_{\substack{1 \le I \le s \\ I \ne i}} \mu_{S_I,R_j \to R_j} \tag{27}$$

the message $\mu_{S_i,R_j \to R_j}$ from the factor $f(S_i, R_j)$ is

$$\mu_{S_i,R_j \to R_j} = \sum_{S_i} f(S_i, R_j)\, \nu_{S_i \to S_i,R_j} \tag{28}$$

the message $\nu_{R_j \to T_j,R_j,X_t}$ to the factor $f(T_j, R_j, X_t)$ is

$$\nu_{R_j \to T_j,R_j,X_t} = \mu_{R_1,R_2,\ldots,R_r,N \to R_j} \prod_{1 \le I \le s} \mu_{S_I,R_j \to R_j} \tag{29}$$

the message $\mu_{T_j,R_j,X_t \to R_j}$ from the factor $f(T_j, R_j, X_t)$ is

$$\mu_{T_j,R_j,X_t \to R_j} = \int f(T_j, R_j, X_t)\, \nu_{X_t \to T_j,R_j,X_t}\, dX_t \tag{30}$$

the message $\nu_{R_j \to R_1,R_2,\ldots,R_r,N}$ to the factor $f(R_1, R_2, \ldots, R_r, N)$ is

$$\nu_{R_j \to R_1,R_2,\ldots,R_r,N} = \mu_{T_j,R_j,X_t \to R_j} \prod_{1 \le I \le s} \mu_{S_I,R_j \to R_j} \tag{31}$$

the message $\mu_{R_1,R_2,\ldots,R_r,N\to R_j}$ from the factor $f(R_1,R_2,\ldots,R_r,N)$ is

$$\mu_{R_1,R_2,\ldots,R_r,N\to R_j} = \tag{32}$$

$$\sum_{\{R_1,\ldots,R_r\}\setminus\{R_j\}} \sum_N f(R_1,\ldots,R_r,N) \tag{33}$$

$$\nu_{N\to R_1,\ldots,R_r,N} \prod_{\substack{1\le J\le r\\ J\ne j}} \nu_{R_J\to R_1,\ldots,R_r,N} \tag{34}$$

For N, the message $\nu_{N\to N}$ to the factor N is

$$\nu_{N\to N} = \mu_{R_1,R_2,\ldots,R_r,N\to N}\mu_{N,T_r\to N} \tag{35}$$

the message $\mu_{N\to N}$ from the factor N is

$$\mu_{N\to N} = \sum_N f(N) \tag{36}$$

the message $\nu_{N\to R_1,R_2,\ldots,R_r,N}$ to the factor $f(R_1,R_2,\ldots,R_r,N)$ is

$$\nu_{N\to R_1,R_2,\ldots,R_r,N} = \mu_{N\to N}\mu_{N,T_r\to N} \tag{37}$$

the message $\mu_{R_1,R_2,\ldots,R_r,N\to N}$ from the factor $f(R_1,R_2,\ldots,R_r,N)$ is

$$\mu_{R_1,R_2,\ldots,R_r,N\to N} = \tag{38}$$

$$\sum_{R_1,\ldots,R_r} f(R_1,\ldots,R_r,N) \prod_{1\le J\le r} \nu_{R_J\to R_1,\ldots,R_r,N} \tag{39}$$

the message $\nu_{N\to N,T_r}$ to the factor $f(N,T_r)$ is

$$\nu_{N\to N,T_r} = \mu_{N\to N}\mu_{R_1,R_2,\ldots,R_r,N\to N} \tag{40}$$

the message $\mu_{N,T_r\to N}$ from the factor $f(N,T_r)$ is

$$\mu_{N,T_r\to N} = \sum_N f(N,T_r) \tag{41}$$

For X_t, the message $\nu_{X_t\to X_{t-1},X_t}$ to the factor $f(X_{t-1},X_t)$ is

$$\nu_{X_t\to X_{t-1},X_t} = \tag{42}$$

$$\mu_{X_t,X_{t+1}\to X_t} \prod_{1\le I\le s} \mu_{S_I,X_t,T_r\to X_t} \prod_{1\le J\le r} \mu_{T_J,R_J,X_t\to X_t} \tag{43}$$

the message $\mu_{X_{t-1},X_t\to X_t}$ from the factor $f(X_{t-1},X_t)$ is

$$\mu_{X_{t-1},X_t\to X_t} = \int f(X_{t-1},X_t)\,\nu_{X_{t-1}\to X_{t-1},X_t}\,dX_{t-1} \tag{44}$$

the message $\nu_{X_t\to X_t,X_{t+1}}$ to the factor $f(X_t,X_{t+1})$ is

$$\nu_{X_t\to X_t,X_{t+1}} = \tag{45}$$

$$\mu_{X_{t-1},X_t\to X_t} \tag{46}$$

$$\prod_{1\le I\le s} \mu_{S_I,X_t,T_r\to X_t} \prod_{1\le J\le r} \mu_{T_J,R_J,X_t\to X_t} \tag{47}$$

the message $\mu_{X_t\to X_t,X_{t+1}}$ from the factor $f(X_t,X_{t+1})$ is

$$\mu_{X_t\to X_t,X_{t+1}} = \int f(X_t,X_{t+1})\,\nu_{X_{t+1}\to X_t,X_{t+1}}\,dX_{t+1} \tag{48}$$

the message $\nu_{X_t\to S_i,X_t,T_r}$ to the factor $f(S_i,X_t,T_r)$ is

$$\nu_{X_t\to S_i,X_t,T_r} = \tag{49}$$

$$\mu_{X_{t-1},X_t\to X_t}\mu_{X_t,X_{t+1}\to X_t} \tag{50}$$

$$\prod_{\substack{1\le I\le s\\ I\ne i}} \mu_{S_I,X_t,T_r\to X_t} \prod_{1\le J\le r} \mu_{T_J,R_J,X_t\to X_t} \tag{51}$$

the message $\mu_{S_i,X_t,T_r\to X_t}$ from the factor $f(S_i,X_t,T_r)$ is

$$\mu_{S_i,X_t,T_r\to X_t} = \sum_{S_i} f(S_i,X_t,T_r)\,\nu_{S_i\to S_i,X_t,T_r} \tag{52}$$

the message $\nu_{X_t\to T_j,R_j,X_t}$ to the factor $f(T_j,R_j,X_t)$ is

$$\nu_{X_t\to T_j,R_j,X_t} = \tag{53}$$

$$\mu_{X_{t-1},X_t\to X_t}\mu_{X_t,X_{t+1}\to X_t} \tag{54}$$

$$\prod_{1\le I\le s} \mu_{S_I,X_t,T_r\to X_t} \prod_{\substack{1\le J\le r\\ J\ne j}} \mu_{T_J,R_J,X_t\to X_t} \tag{55}$$

the message $\mu_{T_j,R_j,X_t\to X_t}$ from the factor $f(T_j,R_j,X_t)$ is

$$\mu_{T_j,R_j,X_t\to X_t} = \sum_{R_j} f(T_j,R_j,X_t)\,\nu_{R_j\to T_j,R_j,X_t} \tag{56}$$

A.2 MESSAGE PASSING FOR HIGH ORDER min FACTORS

Recall that the factor $f_j(T_j,t_1,t_2,\ldots,t_s)$ is given by

$$f_j(T_j,t_1,t_2,\ldots,t_s) = \delta(T_r - t_k) \tag{57}$$

where t_k is the j^{th} minimum element of $\{t_1,t_2,\ldots,t_s\}$. We denote this factor by f_j.

Direct computation of messages in this high order factor graph would require computing an $s-1$-dimensional integral. However, our f_j, which correspond to the j-th minimum function, can be rewritten as a sum of products as,

$$f_j = \sum_{k=1}^{s} \delta(t_k - T_j) \sum_{(A,B)\in\mathcal{S}_k} \prod_{a\in A} \mathbf{1}(t_a < T_j) \prod_{b\in B} \mathbf{1}(t_b > T_j) \tag{58}$$

where $\mathcal{S}_k = \{(A,B)\subseteq[s]\times[s] : A\cup B = [s]\setminus\{k\}, A\cap B = \varnothing, |A| = j-1, |B| = s-j\}$ and $[s] = \{1,2,\ldots,s\}$ The outer sum represents the s different cases where each element of $\{t_1, t_2, \ldots, t_s\}$ can be the j^{th} smallest. Suppose t_k is the j^{th} smallest and is equal to T_j. Then, the remaining $\{t_l | l \neq k\}$ are partitioned into 2 sets, where every t_l in one set is smaller than t_k and while each t_l in the other is larger. There are $\binom{s-1}{j-1}$ such partitions. Thus the f_j corresponds to a sum of products of $O(s\binom{s-1}{j-1})$ terms.

The message $\mu_{f_j \to t_i}(t_i)$ from the factor $f_j (1 \leq j \leq r)$ to the variable $t_i (1 \leq i \leq s)$ is given by:

$$\mu_{f_j \to t_i}(t_i) = \int \left(\prod_{\substack{1 \leq l \leq s \\ l \neq i}} \nu_{t_l \to f_j}(t_l) \right) f(T_j, t_1, t_2, \ldots, t_s) \underbrace{d \ldots t}_{\text{except } dt_i} \tag{59}$$

$$= \int \left(\prod_{\substack{1 \leq l \leq s \\ l \neq i}} \nu_{t_l \to f_j}(t_l) \right) \delta(T_j - t_k) \underbrace{d \ldots t}_{\text{except } dt_i} \tag{60}$$

where t_k is the j^{th} smallest element of $\{t_1, t_2, \ldots, t_s\}$, and $\nu_{t_l \to f_j}(t_l)$ is the message from t_l to f_j.

For computing $\mu_{f_j \to t_i}(t_i)$, f_j can be written as the sum of the following terms:

$$f_j = \delta(t_i - T_j) \sum_{A,B} \prod_{a \in A} \mathbf{1}(t_a < T_j) \prod_{b \in B} \mathbf{1}(t_b > T_j) \tag{61}$$

$$+ \sum_{k \neq i} \delta(t_k - T_j) \sum_{A,B} \prod_{a \in A} \mathbf{1}(t_a < T_j) \prod_{b \in B} \mathbf{1}(t_b > T_j) \tag{62}$$

Then, the multidimensional integral can be written as sum of products of unidimensional integrals. The final computation of the message requires a sum of $O(s\binom{s-1}{j-1})$ terms as,

$$\mu_{f_j \to t_i}(t_i) = \delta(t_i - T_j) h_1(T_j) + \mathbf{1}(t_i < T_j) h_2(T_j) + \mathbf{1}(t_i > T_j) h_3(T_j) \tag{63}$$

where

$$h_1(T_j) = \sum_{A,B} \prod_{a \in A} \left(\int_{-\infty}^{T_j} \nu_{t_a \to f_j}(t_a)\, dt_a \right) \tag{64}$$

$$\prod_{b \in B} \left(\int_{T_j}^{+\infty} \nu_{t_b \to f_j}(t_b)\, dt_b \right) \tag{65}$$

$$h_2(T_j) = \sum_{A,B,i \in A} \prod_{a \in A, a \neq i} \left(\int_{-\infty}^{T_j} \nu_{t_a \to f_j}(t_a)\, dt_a \right) \tag{66}$$

$$\prod_{b \in B} \left(\int_{T_j}^{+\infty} \nu_{t_b \to f_j}(t_b)\, dt_b \right) \tag{67}$$

$$h_3(T_j) = \sum_{A,B,i \in B} \prod_{a \in A} \left(\int_{-\infty}^{T_j} \nu_{t_a \to f_j}(t_a)\, dt_a \right) \tag{68}$$

$$\prod_{b \in B, b \neq i} \left(\int_{T_j}^{+\infty} \nu_{t_b \to f_j}(t_b)\, dt_b \right) \tag{69}$$

For r such factors f_j, if messages are computed directly, each iteration of message passing will require $O(\sum_{j=1}^{r} \binom{s}{j})$ computation. Note that only $2s$ unidimensional integrals need to be computed, and the remainder of the computation corresponds to computing the value of elementary symmetric polynomials, which corresponds to sums of all combinations. To compute a symmetric polynomial $\sum_{\substack{A \in \{1,2,\ldots,n\} \\ |A| = k}} \prod_{a \in A} c_a$ which sums over all k-combinations of $\{c_1, c_2, \ldots, c_n\}$, we can use dynamic programming to find the coefficient of x^k in $\prod_{i=1}^{n} (x + c_i)$, and this can be done in $O(n^2)$ time.

A.2.1 FULL MODEL WITH CLUTTER

We handle two kinds of systematic noise in this model: losses from the sender and clutter. Losses are handled by $m_1, m_2, \ldots, m_s$ in Figure 9.

Clutter can be incorporated in this model through the factor $f'_k(T_k, t_1, t_2, \ldots, t_s, J_1, J_2, \ldots, J_k)$ as

$$f'_k(.) = \begin{cases} \delta(T_r - t_l) & \text{if } J_k = 0 \\ 1 & \text{if } J_k = 1 \end{cases} \tag{70}$$

where t_l is the $(k - \sum_i J_i)^{\text{th}}$ minimum element of $\{t_1, t_2, \ldots, t_s\}$. This is identical to f from the previous section if the J_i are all zero. If $J_k = 1$, i.e. the current message is clutter, then we assume a uniform distribution over T_k. If some previous received message was clutter, T_k will take a lower minimum value.

Then, the factor can be written down in terms of the factors f, from the previous section, as $f'_k(T_k, t_1, t_2, \ldots, t_s, J_1, J_2, \ldots, J_k)$:

$$f'_k(.) = f_{k-\sum_i J_i}(T_k, t_1, t_2, \ldots, t_s) \tag{71}$$

Then, the messages from f_k to t_i can be written as:

$$\nu'_{f'_k \to t_i} = \sum_{J_i} \nu_{f_{k-\sum_i J_i} \to t_i} \pi_l \mu_{J_l \to f'_k} \tag{72}$$

where the summation is over the values 0 or 1 for each J_i.

Messages from f'_k to J_l can be written as:

$$\nu'_{f'_k \to t_i} = \sum_{J_i, i \neq l} \int f_{k-\sum_i J_i} \mathrm{d}t_1 \ldots \mathrm{d}t_s \tag{73}$$

Thus, we can precompute the messages for f_k in polynomial time, and we can compute these messages in $O(2^r)$ additional time.

A.3 ADDITIONAL EXPERIMENTAL RESULTS FOR RAFOS FLOAT DATA

Here we present more additional experimental results for tracking RAFOS floats using our proposed method. When there are at least three actual signal arrival times at each time step, such as float #767 and float #811 (Figure 1), it is possible to estimate a unique track for the float over the entire period of the float's mission (Figure 7 and 8). However, if at some point during a float's mission that there are only two actual signal arrival times for a certain period, then neither using hand labeled data nor our proposed method can uniquely determine the float's location.

An example for float #759 is given here. The signal arrival times for float #759 are shown in Figure 11, where there exists periods of time during float #759's mission when only at most two signal arrivals are available. As shown in Figure 12, we get different results in different runs of the simple particle filter algorithm using hand labeled data (blue), and our proposed algorithm agrees with hand labeled data when there are at least three signal arrival times available.

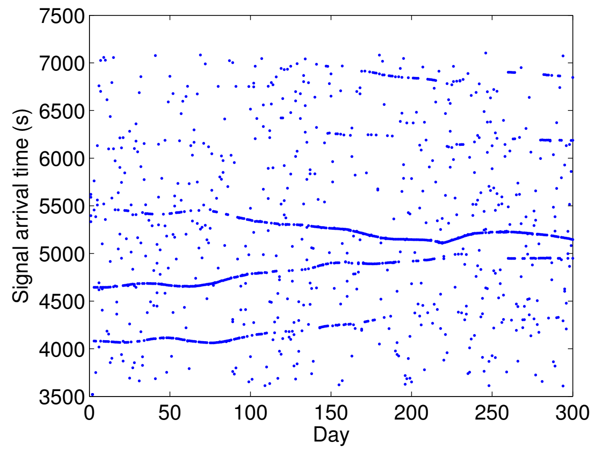
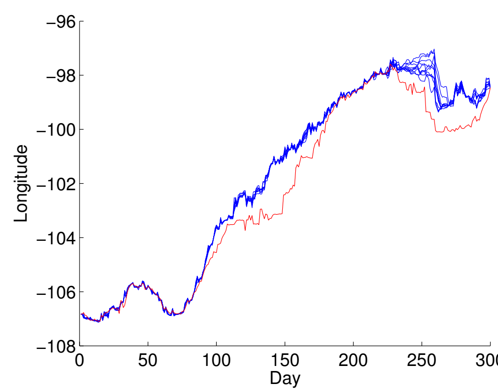
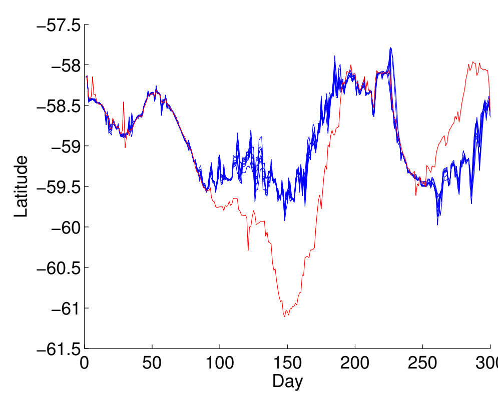


Figure 11: Observed signal arrival times for float #759 over the entire tracking period



(a) Longitude



(b) Latitude

Figure 12: Results of different runs of the simple particle filter algorithm using hand labeled data (blue) versus our proposed algorithm (red) for float #759