Non-parametric causal models

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UAI Tutorial
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Structure

- Part One: Causal DAGs with latent variables
- Part Two: Statistical Models arising from DAGs with latents

Outline for Part One

- Intervention distributions
- The general identification problem
- Tian's ID Algorithm
- Fixing: generalizing marginalizing and conditioning
- Non-parametric constraints aka Verma constraints

Intervention distributions (I)

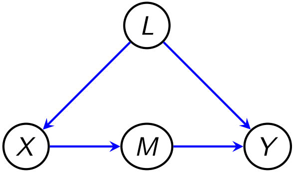
Given a causal DAG $\mathcal{G}$ with distribution:

$$p(V) = \prod_{v \in V} p(v \mid \mathrm{pa}(v))$$

we wish to compute an intervention distribution via truncated factorization:

$$p(V \setminus X \mid \mathrm{do}(X = \mathbf{x})) \quad = \quad \prod_{v \in V \setminus X} p(v \mid \mathrm{pa}(v)).$$

Example



$$p(X, L, M, Y) = p(L)\ p(X \mid L)\ p(M \mid X) p(Y \mid L, M)$$

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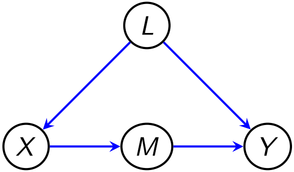

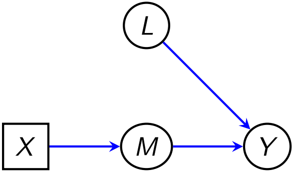

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$$p(L, M, Y \mid \mathrm{do}(X = \tilde{x})) = p(L) \quad \times \quad p(M \mid \tilde{x}) p(Y \mid L, M)$$

Intervention distributions (II)

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Hence if we are interested in $Y \subset V \setminus X$ then we simply marginalize:

$$p(Y \mid \mathrm{do}(X = \mathbf{x})) \quad = \quad \sum_{w \in V \setminus (X \cup Y)} \quad \prod_{v \in V \setminus X} p(v \mid \mathrm{pa}(v)).$$

This is the 'g-computation' formula of Robins (1986).

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Note: $p(Y \mid \mathrm{do}(X = \mathbf{x}))$ is a sum over a product of terms $p(v \mid \mathrm{pa}(v))$.

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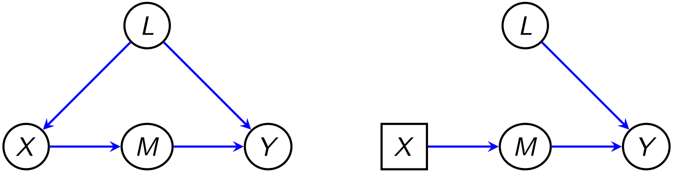

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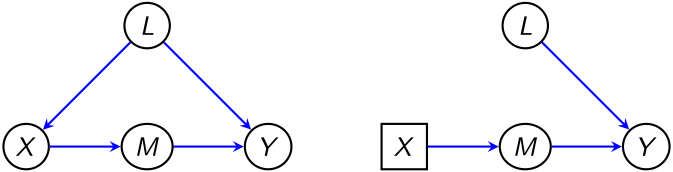

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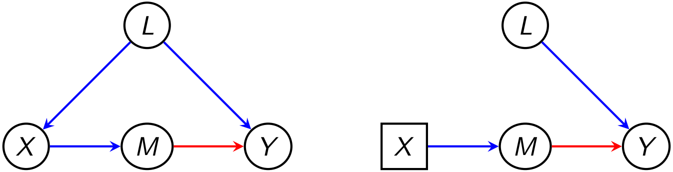
Note that $p(Y \mid \mathrm{do}(X = \tilde{x})) \neq p(Y \mid X = \tilde{x})$.

Example: no effect of M on Y



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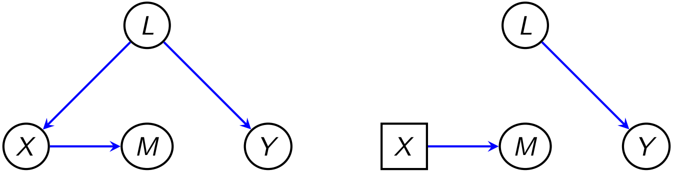

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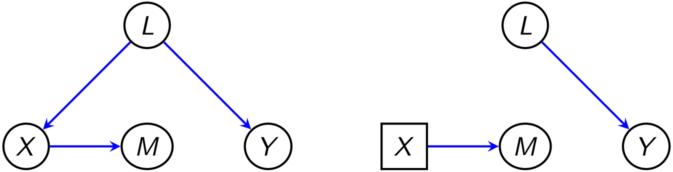

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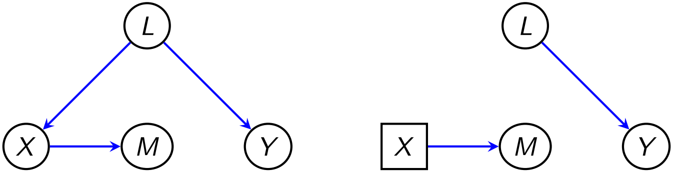

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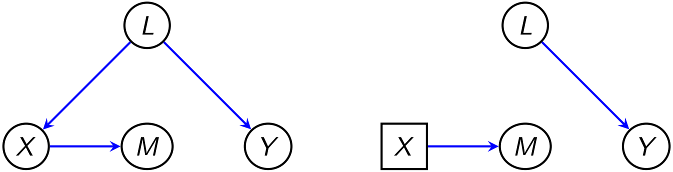

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$$\begin{aligned} p(Y \mid \mathrm{do}(X = \tilde{x})) &= \sum_{l,m} p(L = l)p(M = m \mid \tilde{x})p(Y \mid L = l) \\ &= \sum_{l} p(L = l)p(Y \mid L = l) \end{aligned}$$

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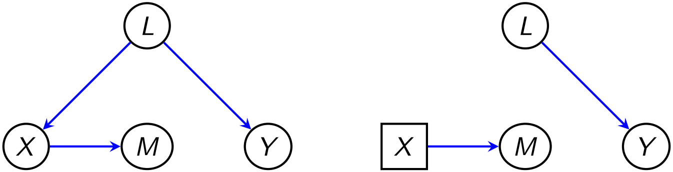

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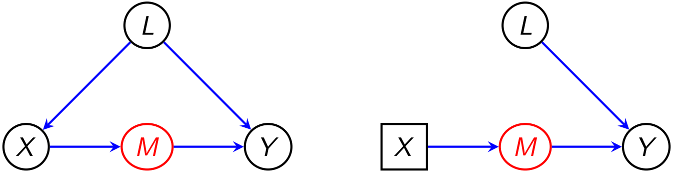
p(Y \mid \mathrm{do}(X = \tilde{x})) &= \sum_{l,m} p(L = l)p(M = m \mid \tilde{x})p(Y \mid L = l) \\
&= \sum_{l} p(L = l)p(Y \mid L = l) \\
&= p(Y) \neq P(Y \mid \tilde{x})
\end{aligned}
$$

since $X \not\!\perp\!\!\!\perp Y$. ‘Correlation is not Causation’.

Example with M unobserved



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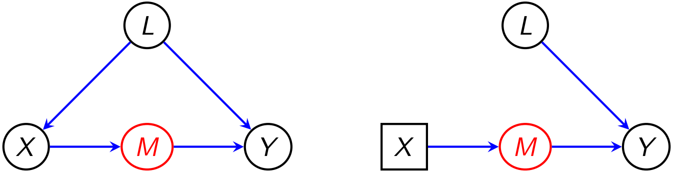
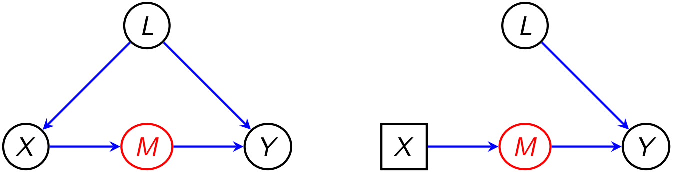


$$p(Y \mid \mathrm{do}(X = \tilde{x})) = \sum_{l,m} p(L = l) p(M = m \mid \tilde{x}) p(Y \mid L = l, M = m)$$
$$= \sum_{l,m} p(L = l) p(M = m \mid \tilde{x}, L = l) p(Y \mid L = l, M = m, X = \tilde{x})$$

Here we have used that $M \perp\!\!\!\perp L \mid X$ and $Y \perp\!\!\!\perp X \mid L, M$.

Example with M unobserved



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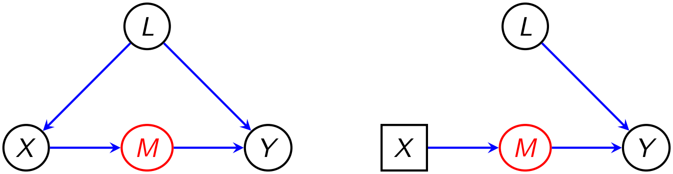

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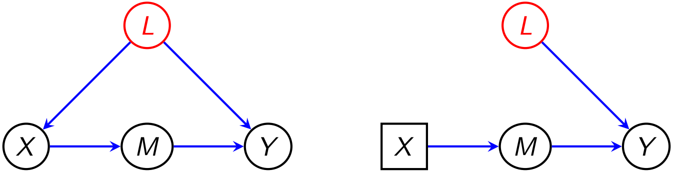
&= \sum_{l} p(L = l) p(Y \mid L = l, X = \tilde{x}).
\end{aligned}
$$

$\Rightarrow$ can find $p(Y \mid \mathrm{do}(X = \tilde{x}))$ even if M not observed.

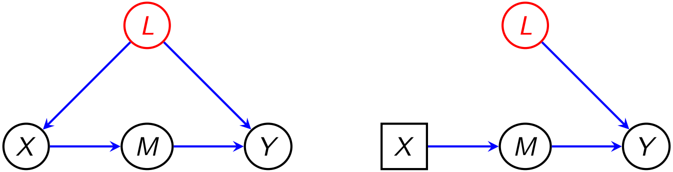
This is an example of the ‘back door formula’.

Example with L unobserved



$p(Y \mid \mathrm{do}(X = \tilde{x}))$

Example with L unobserved



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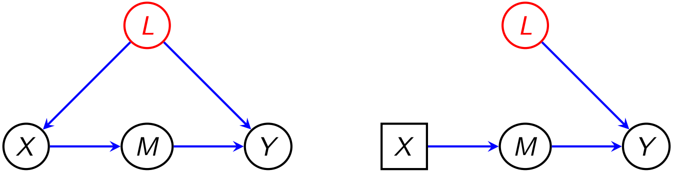
Example with L unobserved



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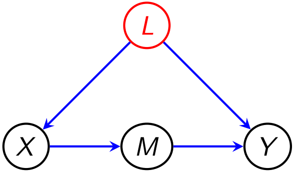

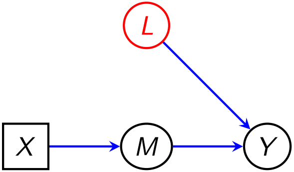

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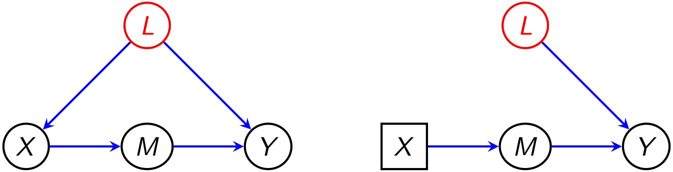
&\quad = \sum_m p(M=m \mid X=\tilde{x}) \left(\sum_{x^*} p(X=x^*)p(Y \mid M=m, X=x^*) \right)
\end{aligned}
$$

Example with L unobserved



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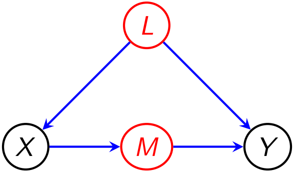
This is an example of the ‘front door formula’.

But with *both* L and M unobserved....



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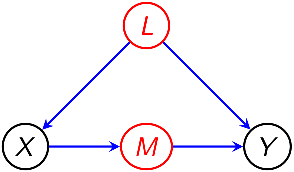
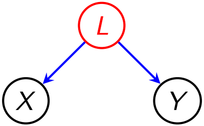


...we are out of luck!

Given $P(X, Y)$, absent further assumptions we cannot distinguish:



General Identification Question

Given: a latent DAG $\mathcal{G}(O \cup H)$, where O are observed, H are hidden, and disjoint subsets $X, Y \subseteq O$.

Q: Is $p(Y \mid \mathrm{do}(X))$ identified given $p(O)$?

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Q: Is $p(Y \mid \text{do}(X))$ identified given $p(O)$?

A: Provide either an identifying formula that is a function of $p(O)$

or report that $p(Y \mid \text{do}(X))$ is not identified.

Latent Projection

Can preserve conditional independences and causal coherence with latents using paths. DAG $\mathcal{G}$ on vertices $V = O \dot{\cup} H$, define **latent projection** as follows: (Verma and Pearl, 1992)

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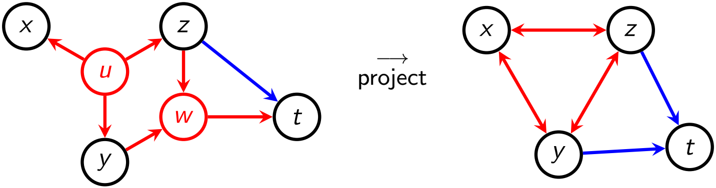
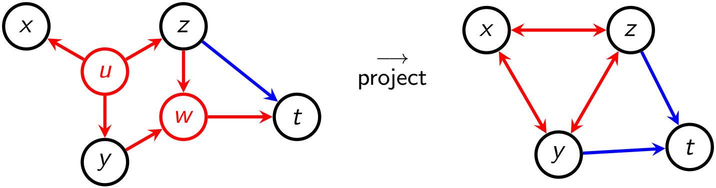
Then remove all latent variables H from the graph.

ADMGs

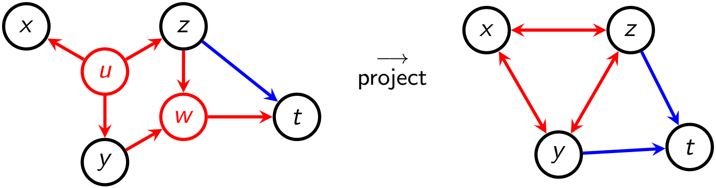


ADMGs



Latent projection leads to an **acyclic directed mixed graph** (ADMG)

ADMGs



Latent projection leads to an **acyclic directed mixed graph** (ADMG)

Can read off independences with d/m-separation.

The projection preserves the causal structure; Verma and Pearl (1992).

‘Conditional’ Acyclic Directed Mixed Graphs

An ‘conditional’ acyclic directed mixed graph (CADMG) is a bi-partite graph $\mathcal{G}(V, W)$, used to represent structure of a distribution over V, indexed by W, for example $P(V \mid \mathrm{do}(W))$.

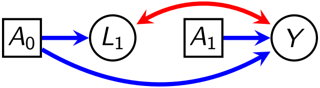
We require:

(i) The induced subgraph of $\mathcal{G}$ on V is an ADMG;

(ii) The induced subgraph of $\mathcal{G}$ on W contains no edges;

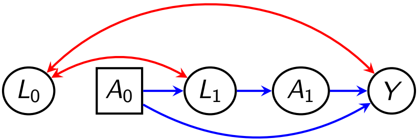
(iii) Edges between vertices in W and V take the form $w \to v$.

We represent V with circles, W with squares:



Here $V = \{L_1, Y\}$ and $W = \{A_0, A_1\}$.

Ancestors and Descendants



In a CADMG $\mathcal{G}(V, W)$ for $v \in V$, let the set of *ancestors* , *descendants* of v be:

$$\mathrm{an}_{\mathcal{G}}(v) = \{a \mid a \to \cdots \to v \text{ or } a = v \text{ in } \mathcal{G}, a \in V \cup W\},$$

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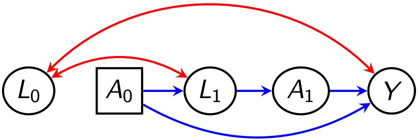

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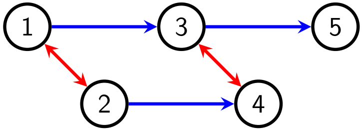
In the example above:

$$\mathrm{an}(y) = \{a_0, l_1, a_1, y\}.$$

Districts

Define a **district** in a C/ADMG to be maximal sets connected by bi-directed edges:



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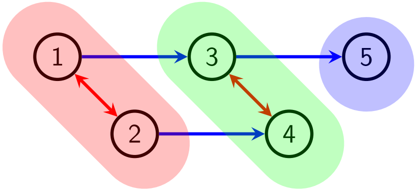

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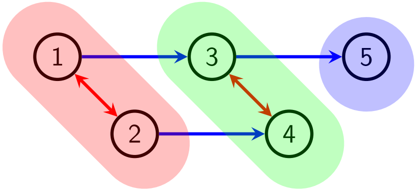

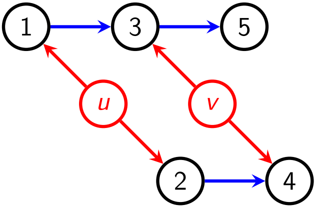

$$\sum_{u,v} p(u)\, p(x_1 \mid u)\, p(x_2 \mid u) \quad p(v)\, p(x_3 \mid x_1, v)\, p(x_4 \mid x_2, v) \quad p(x_5 \mid x_3)$$

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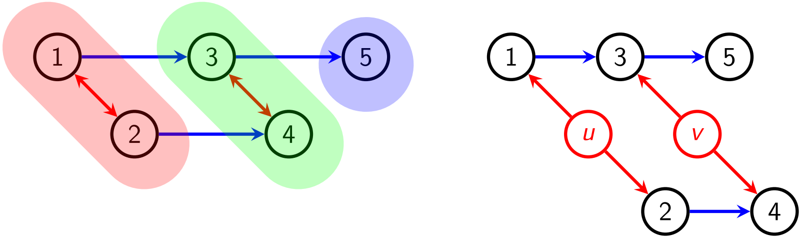

$$\sum_{u,v} \boxed{p(u)\, p(x_1 \mid u)\, p(x_2 \mid u)}\ \boxed{p(v)\, p(x_3 \mid x_1, v)\, p(x_4 \mid x_2, v)}\ \boxed{p(x_5 \mid x_3)}$$

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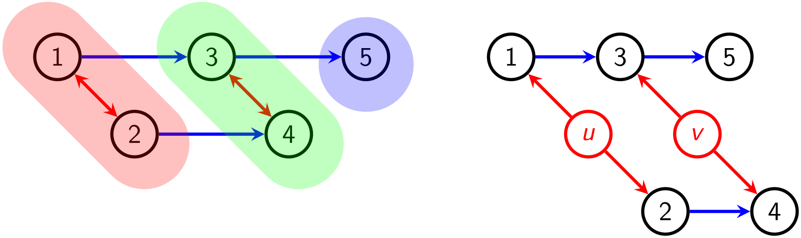

$$\sum_{u,v} p(u)\, p(x_1 \mid u)\, p(x_2 \mid u) \; p(v)\, p(x_3 \mid x_1, v)\, p(x_4 \mid x_2, v) \; p(x_5 \mid x_3)$$

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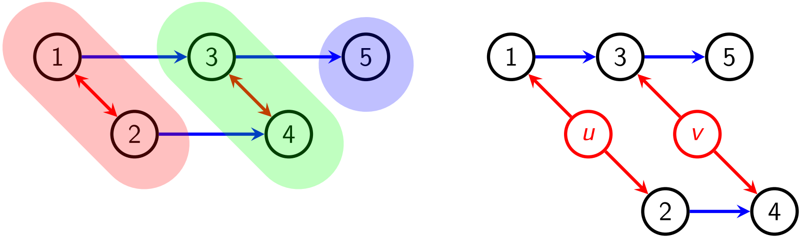

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$$= \boxed{q(x_1, x_2)} \cdot \boxed{q(x_3, x_4 \mid x_1, x_2)} \cdot \boxed{q(x_5 \mid x_3)}.$$

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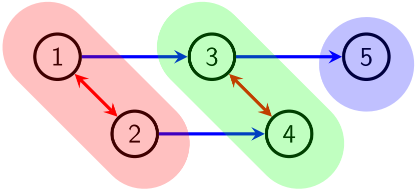

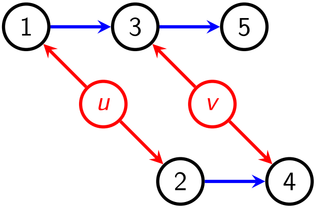

$$
\begin{aligned}
&\sum_{u,v} p(u)\,p(x_1 \mid u)\,p(x_2 \mid u)\; p(v)\,p(x_3 \mid x_1, v)\,p(x_4 \mid x_2, v)\; p(x_5 \mid x_3) \\
&= \sum_{u} p(u)\,p(x_1 \mid u)\,p(x_2 \mid u) \sum_{v} p(v)\,p(x_3 \mid x_1, v)\,p(x_4 \mid x_2, v)\; p(x_5 \mid x_3) \\
&= q(x_1, x_2) \cdot q(x_3, x_4 \mid x_1, x_2) \cdot q(x_5 \mid x_3). \\
&= \prod_i q_{D_i}(x_{D_i} \mid x_{\mathrm{pa}(D_i)\setminus D_i})
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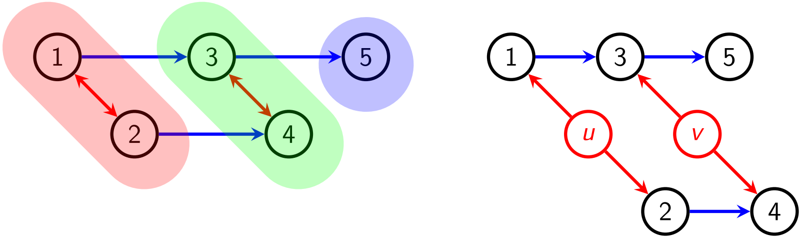

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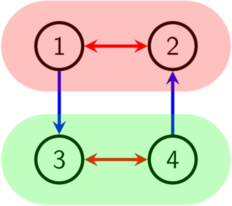
Districts are called ‘c-components’ by Tian.

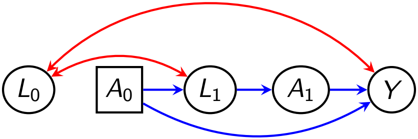
Edges between districts



There is no ordering on vertices such that parents of a district precede every vertex in the district.

(Cannot form a ‘chain graph’ ordering.)

Notation for Districts



In a CADMG $\mathcal{G}(V, W)$ for $v \in V$, the district of v is:

$$\text{dis}_{\mathcal{G}}(v) = \{d \mid d \leftrightarrow \cdots \leftrightarrow v \text{ or } d = v \text{ in } \mathcal{G}, d \in V\}.$$

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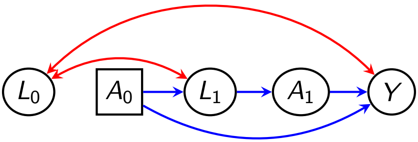

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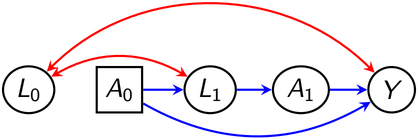

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We use $\mathcal{D}(\mathcal{G})$ to denote the set of districts in $\mathcal{G}$.

In example $\mathcal{D}(\mathcal{G}) = \{\ \{l_0, l_1, y\}, \{a_1\}\ \}$.

Tian's ID algorithm for identifying $P(Y \mid \mathbf{do}(X))$

(A) Re-express the query as a sum over a product of intervention distributions on districts:

$$p(Y \mid \mathrm{do}(X)) = \sum \prod_i p(D_i \mid \mathrm{do}(\mathrm{pa}(D_i) \setminus D_i)).$$

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This is clearly sufficient for identifiability.

Necessity follows from results of Shpitser (2006).

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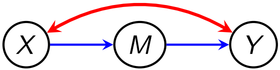
Then:

$$P(Y \mid \mathrm{do}(X)) = \sum_{T \setminus (X \cup Y)} \prod_{D_i} p(D_i \mid \mathrm{do}(\mathrm{pa}(D_i) \setminus D_i)).$$

Example: front door graph

$\mathcal{G}$



$$p(Y \mid \mathrm{do}(X))$$

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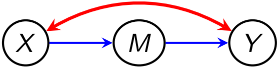

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$\mathcal{G}_{[V \setminus \{X\}]} = \mathcal{G}^*$

$T = \{X, M, Y\}$

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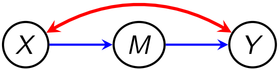

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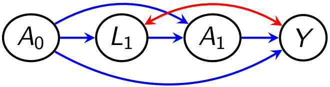
Districts in $T \setminus \{A_0, A_1\}$ are $D_1 = \{M\}$, $D_2 = \{Y\}$.

$$p(Y \mid \mathrm{do}(X)) = \sum_M p(M \mid \mathrm{do}(X)) p(Y \mid \mathrm{do}(M))$$

Example: The Verma Graph

$\mathcal{G}$



$$p(Y \mid \mathrm{do}(A_0, A_1))$$

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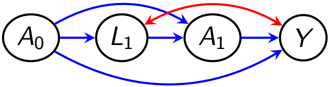

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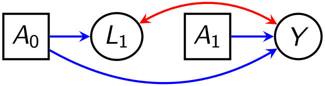
$\mathcal{G}_{[V\setminus\{A_0,A_1\}]}$



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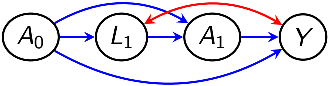
Example: The Verma Graph

$\mathcal{G}$



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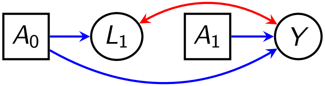

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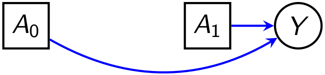

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Example: The Verma Graph

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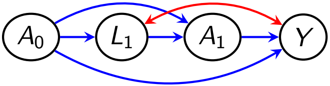
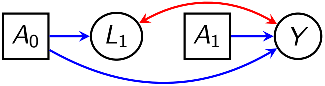


$p(Y \mid \mathrm{do}(A_0, A_1))$

$\mathcal{G}_{[V \setminus \{A_0, A_1\}]}$



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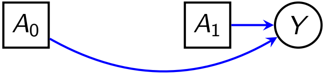

$D_1 = \{Y\}$

(Here the decomposition is trivial since there is only one district and no summation.)

(B) Finding if $P(D \,|\, \mathbf{do}(\mathrm{pa}(D) \setminus D))$ is identified

Idea: Find an ordering $r_1, \ldots, r_p$ of $O \setminus D$ such that:

If $P(O \setminus \{r_1, \ldots, r_{t-1}\} \,|\, \mathrm{do}(r_1, \ldots, r_{t-1}))$ is identified

Then $P(O \setminus \{r_1, \ldots, r_t\} \,|\, \mathrm{do}(r_1, \ldots, r_t))$ is also identified.

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Sufficient for identifiability of $P(D \,|\, \mathrm{do}(\mathrm{pa}(D) \setminus D))$, since:

$P(O)$ is identified

$D = O \setminus \{r_1, \dots, r_p\}$, so
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Such a vertex r_t will said to be 'fixable', given that we have already 'fixed' $r_1, \ldots, r_{t-1}$:

'fixing' differs from 'do'/intervening since the latter does not preserve identifiability.

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To do:

- Give a graphical characterization of 'fixability';
- Construct the identifying formula.

The set of fixable vertices

Given a CADMG $\mathcal{G}(V, W)$ we define the set of fixable vertices,

$$F(\mathcal{G}) \equiv \{v \mid v \in V, \mathrm{dis}_{\mathcal{G}}(v) \cap \mathrm{de}_{\mathcal{G}}(v) = \{v\}\}.$$

In words, a vertex $v \in V$ is fixable in $\mathcal{G}$ if there is no (proper) descendant of v that is in the same district as v in $\mathcal{G}$.

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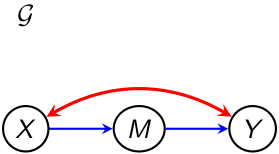
Thus v is fixable if there is no vertex $y \neq v$ such that

$$v \leftrightarrow \cdots \leftrightarrow y \quad \text{and} \quad v \rightarrow \cdots \rightarrow y \quad \text{in } \mathcal{G}.$$

Note that the set of fixable vertices is a subset of V, and contains at least one vertex from each district in $\mathcal{G}$.

Example: front door graph

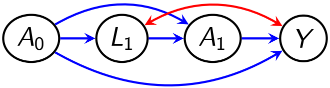


$F(\mathcal{G}) = \{M, Y\}$

X is not fixable since Y is a descendant of X and

Y is in the same district as X

Example: The Verma Graph



Here $F(\mathcal{G}) = \{A_0, A_1, Y\}$.

L_1 is not fixable since Y is a descendant of L_1 and

Y is in the same district as L_1.

The *graphical* operation of fixing vertices

Given a CADMG $\mathcal{G}(V, W, E)$, for every $r \in F(\mathcal{G})$ we associate a transformation ϕ_r on the pair $(\mathcal{G}, P(X_V \mid X_W))$:

$$\phi_r(\mathcal{G}) \equiv \mathcal{G}^{\dagger}(V \setminus \{r\}, W \cup \{r\}),$$

where in $\mathcal{G}^{\dagger}$ we remove from $\mathcal{G}$ any edge that has an arrowhead at r.

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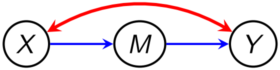
The operation of 'fixing r' simply transfers r from 'V' to 'W', and removes edges $r \leftrightarrow$ or $r \leftarrow$.

Example: front door graph

$\mathcal{G}$



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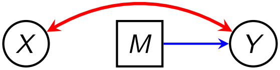
$\phi_M(\mathcal{G})$

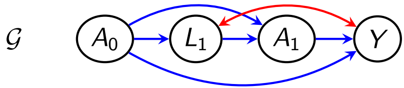


$F(\phi_M(\mathcal{G})) = \{X, Y\}$

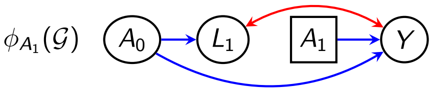
Note that X was not fixable in $\mathcal{G}$,

but it is fixable in $\phi_M(\mathcal{G})$ after fixing M.

Example: The Verma Graph



Here $F(\mathcal{G}) = \{A_0, A_1, Y\}$.



Notice $F(\phi_{A_1}(\mathcal{G})) = \{A_0, L_1, Y\}$.

Thus L_1 was not fixable prior to fixing A_1,

but L_1 is fixable in $\phi_{A_1}(\mathcal{G})$ after fixing A_1.

The *probabilistic* operation of fixing vertices

Given a distribution $P(V \mid W)$ we associate a transformation:

$$\phi_r(P(V \mid W); \mathcal{G}) \quad \equiv \quad P(V \mid W)/P(r \mid \mathrm{mb}_{\mathcal{G}}(r)).$$

Here
$\mathrm{mb}_{\mathcal{G}}(r) = \{y \neq r \mid (r \leftarrow y) \text{ or } (r \leftrightarrow \circ \cdots \circ \leftrightarrow y) \text{ or } (r \leftrightarrow \circ \cdots \circ \leftrightarrow \circ \leftarrow y)\}$.

In words: *we divide by the conditional distribution of r given the other vertices in the district containing r, and the parents of the vertices in that district.*

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In words: *we divide by the conditional distribution of r given the other vertices in the district containing r, and the parents of the vertices in that district.*

It can be shown that if r is fixable in $\mathcal{G}$ then:

$$\phi_r(P(V \mid \mathrm{do}(W)); \mathcal{G}) = P(V \setminus \{r\} \mid \mathrm{do}(W \cup \{r\})).$$

as required.

Note: If r is fixable in $\mathcal{G}$ then $\mathrm{mb}_{\mathcal{G}}(r)$ is the 'Markov blanket' of r in $\mathrm{an}_{\mathcal{G}}(\mathrm{dis}_{\mathcal{G}}(r))$.

Unifying Marginalizing and Conditioning

Some special cases:

- If $\mathrm{mb}_{\mathcal{G}}(r) = (V \cup W) \setminus \{r\}$ then fixing corresponds to marginalizing:

$$\phi_r(P(V \mid W); \mathcal{G}) = \frac{P(V \mid W)}{P(r \mid (V \cup W) \setminus \{r\})} = P(V \setminus \{r\} \mid W)$$

- If $\mathrm{mb}_{\mathcal{G}}(r) = W$ then fixing corresponds to ordinary conditioning:

$$\phi_r(P(V \mid W); \mathcal{G}) = \frac{P(V \mid W)}{P(r \mid W)} = P(V \setminus \{r\} \mid W \cup \{r\})$$

- In the general case fixing corresponds to re-weighting, so

$$\phi_r(P(V \mid W); \mathcal{G}) = P^*(V \setminus \{r\} \mid W \cup \{r\}) \neq P(V \setminus \{r\} \mid W \cup \{r\})$$

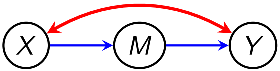
Composition of fixing operations

We use $\circ$ to indicate composition of operations in the natural way, so that:

$$\begin{aligned} \phi_r \circ \phi_s(\mathcal{G}) &\equiv \phi_r(\phi_s(\mathcal{G})) \\ \phi_r \circ \phi_s(P(V \mid W); \mathcal{G}) &\equiv \phi_r\left(\phi_s\left(P(V \mid W); \mathcal{G}\right); \phi_s(\mathcal{G})\right) \end{aligned}$$

Example: front door graph (D_1)

$\mathcal{G}$



$F(\mathcal{G}) = \{M, Y\}$

$\phi_Y(\mathcal{G})$

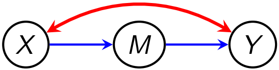
$F(\phi_Y(\mathcal{G})) = \{X, M\}$

$\phi_X \circ \phi_Y(\mathcal{G})$

This proves that $p(M \mid \text{do}(X))$ is identified.

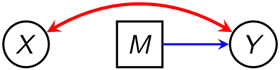
Example: front door graph (D_2)

$\mathcal{G}$



$F(\mathcal{G}) = \{M, Y\}$

$\phi_M(\mathcal{G})$



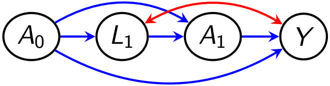
$F(\phi_M(\mathcal{G})) = \{X, Y\}$

$\phi_X \circ \phi_M(\mathcal{G})$

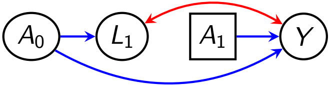
This proves that $p(Y \mid \mathrm{do}(M))$ is identified.

Example: The Verma Graph

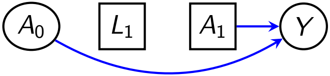
$\mathcal{G}$



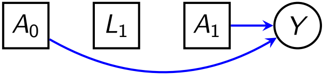
$\phi_{A_1}(\mathcal{G})$



$\phi_{L_1} \circ \phi_{A_1}(\mathcal{G})$



$\phi_{A_0} \circ \phi_{L_1} \circ \phi_{A_1}(\mathcal{G})$



This establishes that $P(Y \mid \mathrm{do}(A_0, A_1))$ is identified.

Review: Tian's ID algorithm via fixing

(A) Re-express the query as a sum over a product of intervention distributions on districts:

$$p(Y \mid \text{do}(X)) = \sum \prod_i p(D_i \mid \text{do}(\text{pa}(D_i) \setminus D_i)).$$

- Cut edges into X;
- Restrict to vertices that are (still) ancestors of Y;
- Find the set of districts $D_1, \ldots, D_p$.

Review: Tian's ID algorithm via fixing

(A) Re-express the query as a sum over a product of intervention distributions on districts:

$$p(Y \mid \mathrm{do}(X)) = \sum \prod_i p(D_i \mid \mathrm{do}(\mathrm{pa}(D_i) \setminus D_i)).$$

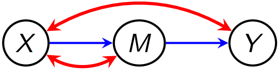
- Cut edges into X;
- Restrict to vertices that are (still) ancestors of Y;
- Find the set of districts $D_1, \ldots, D_p$.

(B) Check whether each term: $p(D_i \mid \mathrm{do}(\mathrm{pa}(D_i) \setminus D_i))$ is identified.

- Iteratively find a vertex that r_t that is fixable in $\phi_{r_{t-1}} \circ \cdots \circ \phi_{r_1}(\mathcal{G})$, with $r_t \notin D_i$;
- If no such vertex exists then $P(D_i \mid \mathrm{do}(\mathrm{pa}(D_i) \setminus D_i))$ is not identified.

Not identified example

$\mathcal{G}$



$F(\mathcal{G}) = \{Y\}$

We see that $p(Y \mid \mathrm{do}(M))$ is not identified

since the only fixable vertex is Y.

Reachable subgraphs of an ADMG

A CADMG $\mathcal{G}(V, W)$ is *reachable* from ADMG $\mathcal{G}^*(V \cup W)$ if there is an ordering of the vertices in $W = \langle w_1, \ldots, w_k \rangle$, such that for $j = 1, \ldots, k$,

$$w_1 \in F(\mathcal{G}^*) \text{ and for } j = 2, \ldots, k,$$
$$w_j \in F(\phi_{w_{j-1}} \circ \cdots \circ \phi_{w_1}(\mathcal{G}^*)).$$

Thus a subgraph is reachable if, under some ordering, each of the vertices in W may be fixed, first in $\mathcal{G}^*$, and then in $\phi_{w_1}(\mathcal{G}^*)$, then in $\phi_{w_2}(\phi_{w_1}(\mathcal{G}^*))$, and so on.

Intrinsic sets

A set D is said to be *intrinsic* if it forms a *district* in a *reachable* subgraph.

If D is intrinsic in $\mathcal{G}$ then $p(D \mid \text{do}(\text{pa}(D) \setminus D))$ is identified.

The intervention distributions $p(D \mid \text{do}(pa(D) \setminus D))$ for intrinsic D play the same role as $P(v \mid \text{do}(\text{pa}(v))) = p(v \mid \text{pa}(v))$ in the simple fully observed case.

Given an ADMG $\mathcal{G}$ we let $\mathcal{I}(\mathcal{G})$ denote the intrinsic sets in $\mathcal{G}$.

Intrinsic sets and 'hedges'

Shpitser (2006) provided a characterization in terms of graphical structures called 'hedges' of those interventional distributions that were *not* identified.

It may be shown that if a $\leftrightarrow$-connected set is *not* intrinsic then there exists a hedge, hence we have:

$\leftrightarrow$-connected set S is intrinsic iff $p(S \mid \mathrm{do}(\mathrm{pa}(S) \setminus S))$ is identified.

It follows that intrinsic sets may thus also be defined in terms of the *non-existence* of a hedge.

Deriving constraints via fixing

Let $p(O)$ be the observed margin from a DAG with latents $\mathcal{G}(O \cup H)$,

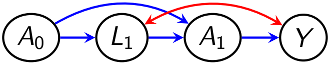
Idea: If $r \in O$ is fixable then $\phi_r(p(O); \mathcal{G})$ will obey the Markov property for the graph $\phi_r(\mathcal{G})$.

...and this can be iterated.

This gives non-parametric constraints that are not independences, that are implied by the latent DAG.

Example: The Verma Constraint

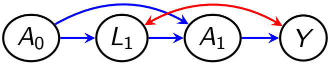
$\mathcal{G}$



Here $F(\mathcal{G}) = \{A_0, A_1, Y\}$.

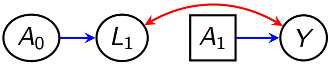
Example: The Verma Constraint

$\mathcal{G}$



Here $F(\mathcal{G}) = \{A_0, A_1, Y\}$.

$\phi_{A_1}(\mathcal{G})$



$$
\begin{array}{rcl}
\phi_{A_1}(p(A_0, L_1, A_1, Y)) & = & p(A_0, L_1, A_1, Y)/p(A_1 \mid A_0, L_1) \\
A_0 \perp\!\!\!\perp Y \mid A_1 & & [\phi_{A_1}(p(A_0, L_1, A_1, Y); \mathcal{G})]
\end{array}
$$

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