We now show that the proposed slice sampler defines a valid Markov chain Monte Carlo algorithm (Theorem 0.2). In particular, (1) the exact posterior \( \pi \) is the invariant distribution of the Markov chain, and (2) that a law of large numbers holds: for any measurable function \( \Phi \) and initial state \( S_0 \), the sequence of states \( S_1, S_2, \ldots \) produced by the slice sampler satisfies

\[
\frac{1}{T} \sum_{t=1}^{T} \Phi(S_t) \overset{a.s.}{\to} \mathbb{E}_\pi [\Phi(S)].
\]

We start with some basic notation. Let \( S \) be a set endowed with a \( \sigma \)-algebra \( \mathcal{B} \), and let \( \pi \) be a target probability distribution on \( S \). A Markov kernel \( \kappa : S \times \mathcal{B} \to [0, 1] \) satisfies two properties: (1) for each \( B \in \mathcal{B} \), \( \kappa(\cdot, B) : S \to [0, 1] \) is a measurable function, and (2) for each \( s \in S \), \( \kappa(s, \cdot) \) is a probability measure. \( \kappa(s, B) \) can be thought of as the probability of transitioning to any state \( s' \in B \subseteq S \) in a single jump starting from a particular state \( s \in S \).

Given two Markov kernels \( \kappa_1, \kappa_2 \), define the composition \( \kappa_1 \circ \kappa_2 \) of the kernels—another Markov kernel—via

\[
(\kappa_1 \circ \kappa_2)(s, B) = \int \kappa_1(s', B) \kappa_2(s, ds').
\]

As with a single kernel, the composition \( (\kappa_1 \circ \kappa_2)(s, B) \) can be thought of as the probability of transitioning to any state \( s' \in B \subseteq S \) after two jumps—first via \( \kappa_2 \), then via \( \kappa_1 \)—starting from a particular state \( s \in S \).

One of the key conditions for a kernel \( \kappa \) to create a Markov chain Monte Carlo scheme for a target distribution \( \pi \) is \( \pi \)-invariance: if one samples \( s \sim \pi \), and then simulates a transition \( s' \sim \kappa(s, \cdot) \), we require that \( s' \sim \pi \). In other words, for any measurable set \( B \),

\[
\int \kappa(s, B) \pi(ds) = \pi(B).
\]

We use the following results in Lemma 0.1 to analyze the \( \pi \)-invariance of the proposed slice sampler for the posterior distribution \( \pi \).

**Lemma 0.1.** Let \( (\kappa_j)_{j=1}^\infty \) be Markov kernels, and suppose \( S \) can be written as a countable partition \( S = \bigcup_j B_j \), \( i \neq j \implies B_i \cap B_j = \emptyset \) of sets of nonzero measure \( \pi(B_j) > 0 \).

1. If the \( \kappa_j \) are all \( \pi \)-invariant, and

\[
\kappa(s, B) = \lim_{J \to \infty} (\kappa_J \circ \cdots \circ \kappa_1)(s, B)
\]

exists pointwise for \( s \in S \) and \( B \in \mathcal{B} \), then \( \kappa \) is a \( \pi \)-invariant Markov kernel.

2. If each \( \kappa_j \) is \( \pi_j \)-invariant, where

\[
\pi_j(B) = \frac{\pi(B \cap B_j)}{\pi(B_j)}
\]

then

\[
\kappa(s, B) = \lim_{J \to \infty} \sum_{j=1}^\infty \mathbb{1}(s \in B_j) \kappa_j(s, B)
\]

is \( \pi \)-invariant.

**Proof.** For 1,

\[
\int \kappa(s, B) \pi(ds) = \lim_{J \to \infty} \int \kappa_J(s, B) \pi(ds) = \lim_{J \to \infty} \int (\kappa_J \circ \cdots \circ \kappa_1)(s, B) \pi(ds) = \lim_{J \to \infty} \pi(B) = \pi(B),
\]

where we use the fact that the finite composition of \( \pi \)-invariant kernels is \( \pi \)-invariant e.g. by [1, p. 49], and Lebesgue dominated convergence to swap the limit and...
We have now shown that the Markov kernel created by

\[
\int \kappa(s, B) \pi(ds) = \sum_{j=1}^{\infty} \int 1_{[s \in B_j]} \kappa_j(s, B) \pi(ds)
\]

where we again use Lebesgue dominated convergence to

complete the final result in Theorem 0.2.

The only remaining kernel is \(\kappa\), the slice sampler in the main text is \(\pi\)-invariant, where \(\pi\) is the posterior conditioned on \(K_{\text{prev}} = j\), which follows from the fact that \(\pi\) is a Gibbs kernel.

We have now shown that the Markov kernel created by the slice sampler in the main text is \(\pi\)-invariant. We now complete the final result in Theorem 0.2.

**Theorem 0.2.** If \(f > 0\) and \(h > 0\), then for any measurable function \(\Phi\) and any initial random state \(S_0\), the sequence of states \(S_1, S_2, \ldots\) produced by \(\kappa\) satisfies

\[
\frac{1}{T} \sum_{t=1}^{T} \Phi(S_t) \xrightarrow{a.s.} E_\pi[\Phi(S)].
\]

**Proof.** We first establish \(\varphi\)-irreducibility: let us set \(\varphi\) to the posterior distribution, let \(s = (v, \gamma, x, \psi, u)\) denote an initial state, and \(B\), a target set of configurations with positive posterior probability. It may not be possible to go from \(s\) to \(B\) in one application of \(\kappa\) as the current configuration of the matrix \(x\) constrains what values \(u\) can take. However this obstacle disappears by considering paths obtained by two applications of \(\kappa\) and visiting an intermediate state where every entry in the matrix \(x\) is set to zero. To formalize this, let \(B_0 = \{(v, \gamma, x, \psi, u) : x_{nk} = 0 \forall n, k\}\). Then

\[
\kappa^2(s, B) = \int \kappa(s, ds') \kappa(s', B) \geq \int \mu(ds') \kappa(s', B)
\]

where \(\mu(A) = \kappa(s, A \cap B_0)\). Using the fact that \(\xi\) is monotonically decreasing, our assumption that \(f\) and \(h\) are strictly positive, we obtain from the full conditional of \(X\) derived in the paper that \(\mu\) is a strictly positive measure on \(B_0\). Moreover, using again the same assumptions, straightforward checks on each full conditional derived in the paper shows that provided \(s \in B_0\), the function \(\kappa(s', B)\) is positive.

Having established \(\varphi\)-irreducibility, Harris recurrence follows from [2 Cor. 13] since \(\kappa\) is a deterministic alternation of Gibbs kernels. Therefore the law of large number follows by [3 Thm. 17.0.1, 17.1.6].

**References**

