8 Appendix: Proofs

Lemma 4. Suppose $\Lambda$ is a deterministic uniformly least favorable distribution for composite vs. simple test ($H_0$ vs. $h_1$) under $\mathcal{M} = (\mathcal{S}, \Theta, \bar{\pi})$. Then for any $n \in \mathbb{N}$, $\Lambda$ is also a uniformly least favorable distribution for testing $H_0$ vs. $h_1$ under $\mathcal{M} = (\mathcal{S}^n, \Theta, \bar{\pi})$ with $n$ i.i.d. samples.

Proof: Let $\text{Spt}(\Lambda) = \{h_0^n\}$. For any $n \in \mathbb{N}$ and any $h_0 \in H_0$, we define a random variable $X_{n,h_0} : \mathcal{S}^n \to \mathbb{R}$, where for any $P_n \in \mathcal{S}^n$, $\text{Pr}(P_n) = \pi_{h_0}(P_n) = \prod_{V \in P_n} \pi_{h_0}(V)$, and $X_{n,h_0}(P_n) = \log \text{Ratio}_{h_0^n,h_1}$. It follows that

$$X_{n,h_0} = X_{h_0} + X_{h_0} + \cdots + X_{h_0}$$

By Lemma 3, for any $h_0 \in H_0$, $X_{h_0^n}$ weakly dominates $X_{h_0}$. Because first-order stochastic dominance is preserved under convolution [Deelstra and Plantin, 2014], we have that $X_{n,h_0}$ weakly dominates $X_{n,h_0}$. The lemma follows after applying Lemma 3.

Remarks. Lemma 4 is an extension of Theorem 2.3 by Reinhardt Reinhardt [1961] to finite models. Reinhardt's theorem requires that for any constant $t$, with measure 0 we have $\pi_{h_0^n}(P) = t \pi_{h_1}(P)$. This is an important assumption in Reinhardt's proof because it assumes away cases with $\text{Ratio}(P) = k_\alpha$ so that the most powerful test is deterministic. Unfortunately, this assumption does not hold for finite models and we must deal with randomized tests.

Lemma 5 Under a Mallows’ model, for any $\varphi$, any $K \in \mathbb{N}$, any $a \in A$, any $W \in \mathcal{L}(A)$, and any $C^0, C \subseteq A$ such that $C$ dominates $C^0$ w.r.t. $W$, we have $\forall_{\varphi}(\{P : \text{w}_P(C^0 \succ a) \geq K\}) \leq \forall_{\varphi}(\{P : \text{w}_P(C \succ a) \geq K\})$.

Proof: We first prove the lemma for a special case where $C$ and $C^0$ differ in only one alternative, that is, $|C - C^0| = 1$. Let $c \in C$ such that $c \notin C^0$. Let $c' \in C^0$ such that $c' \notin C$. Because $C$ dominates $C^0$ in $W$, we have $c \succ_W c'$. Let $P = \{P \in \mathcal{L}(A) : \text{w}_P(C \succ a) \geq K\}$ and $P' = \{P \in \mathcal{L}(A) : \text{w}_P(C^0 \succ a) \geq K\}$. We define the following permutation $\mathcal{M}$ over $\mathcal{L}(A)$. For any $P \in \mathcal{L}(A)$, if $c \succ_P a \succ_P c'$ then $\mathcal{M}(P)$ is the ranking that is obtained from $P$ by switching $c$ and $c'$; otherwise $\mathcal{M}(P) = P$. Because $|C - C^0| = 1$, it follows that for any $P \in P - P'$, we must have $c \succ_P a \succ_P c'$ and $(C - C^0)^0 \succ_P a$. Therefore, $\mathcal{M}(P - P') = P' - P$.

We now prove that $\forall_{\varphi}(P - P') > \forall_{\varphi}(P' - P)$. For any $P \in P - P'$, we have $c \succ_P a \succ_P c'$, which means that $\forall_{\varphi}(P) \geq \forall_{\varphi}(\mathcal{M}(P))/\varphi$ because $c \succ_W c'$. Therefore, $\forall_{\varphi}(P - P') > \forall_{\varphi}(P' - P)$ because $\mathcal{M}(P - P') = P' - P$.

We have $\forall_{\varphi}(P) = \forall_{\varphi}(P \cap P') + \forall_{\varphi}(P - P') \geq \forall_{\varphi}(P \cap P') + \forall_{\varphi}(P' - P) = \forall_{\varphi}(P')$. Therefore, the lemma holds for the case where $|C - C^0| = 1$. For general $C$ and $C^0$, because $C$ dominates $C^0$, there exists a sequence of sets $C = C_0, C_1, \ldots, C_l = C^0$ such that for all $0 \leq i \leq l - 1$, (i) $C_i$ dominates $C_{i+1}$; (ii) $|C_i - C_{i+1}| = 1$. It follows that $\forall_{\varphi}(\{P : \text{w}_P(C \succ a) \geq K\}) \geq \forall_{\varphi}(\{P : \text{w}_P(C_1 \succ a) \geq K\}) \geq \cdots \geq \forall_{\varphi}(\{P : \text{w}_P(C^0 \succ a) \geq K\})$.

Theorem 2 (Characterization of all UMP non-winner tests under Mallows). Given a Mallows’ model $\mathcal{M}_{\text{Ma}}$ with $m \geq 2$ and $n \geq 2$, there exists a UMP test for $H_0 = L_{\alpha}\text{-others}$ vs. $H_1$ for all $0 < \alpha < 1$ if and only if there exists $B \subseteq A$ such that $H_1 \subsetneq L_{B^\alpha}.$

Moreover, when $H_1 \subset L_{B^\alpha}$, $f_{\alpha,B}$ defined in Theorem 1 is a UMP test.

Proof: The “if” part. We note that $f_{\alpha,B}$ does not depend on the orderings among alternatives in $B$ in $h_1$. It follows that for all $h_1 \in H_1$, $f_{\alpha,B}$ is a level-$\alpha$ most powerful test for $H_0$ vs. $\{h_1\}$, which means that $f_{\alpha,B}$ is a UMP test.

The “only if” part. Suppose there exist $B, B'$ such that $B \neq B'$ and there exist two rankings $h_1^B = [B^\alpha \succ a \succ \text{others}]$ and $h_1^{B'} = [B'^\alpha \succ a \succ \text{others}]$ in $H_1$. W.l.o.g. suppose $B' - B \neq \emptyset$. Let $a$ denote the number such that $K_a = n|B| - 0.5$, $\Gamma_a = 0$, and let $f_{\alpha,B}$ denote the most powerful test for $H_0$ vs. $h_1^B$ guaranteed by Theorem 1. Because $K_a$ is not an integer, there does not exist $P_a$ such that $w_{P_a}(B \succ a) > K_a$. This means that $f_{\alpha,B}$ is the unique most powerful level-$\alpha$ test for $H_0$ vs. $h_1^B$. We observe that for any $P_n$, $f_{\alpha,B}(P_n)$ is either 0 or 1, and $f_{\alpha,B}(P_n) = 1$ if and only if $a$ is ranked below $B$ in all $n$ rankings in $P_n$. It follows that $f_{\alpha,B}$ must be the unique level-$\alpha$ UMP test for $H_0$ vs. $H_1$. 
By Theorem 1, any most powerful level-α test, in particular $f_{\alpha, a, B}$, must agree with $f_{\alpha, a, B'}$ except for the threshold cases $w_{P_n'}(B' \succ a) = K'_a$ for some $K'_a$. Choose arbitrary $b' \in B' - B$ and $b \in B$. Let $P_n'$ be composed of $n$ copies of $[b' \succ a \succ \text{others}]$ and let $P_n''$ be composed of $n-1$ copies of $[b' \succ B \succ a \succ \text{others}]$ and one copy of $[b' \succ (B \setminus \{b\}) \succ a \succ \text{others}]$. Because $w_{P_n''}(B \succ a) = n|B| > K_\alpha$, we have $f_{\alpha, a, B}(P_n'') = 1$. This means that the threshold $K'_a$ for $f_{\alpha, a, B}$ is no more than when $w_{P_n'}(B' \succ a) = n|B \cap B'|$. Because $n \geq 2$, we have $w_{P_n'}(B' \succ a) \geq n(|B \cap B'| + 1) \geq n|B \cap B'| = w_{P_n''}(B' \succ a)$, which means that $f_{\alpha, a, B}(P_n'') = 1$. However, $w_{P_n''}(B \succ a) = n|B| - 1 < n|B|$, which is a contradiction because for any profile $P_n$, $f_{\alpha, a, B}(P_n) = 1$ if and only if $B \succ a$ in all $n$ rankings in $P_n$.

\[\square\]

**Theorem 4.** Let $\mathcal{M}^M_a$ denote a Mallows’ model with $n = 1$, any $m \geq 4$, and any $\varphi < 1/m$. There exists $0 < \alpha < 1$ such that no level-α UMP test exists for $H_0 = (\mathcal{L}(A) - H_1)$ vs. $H_1 = L_{a \succ \text{others}}$.

**Proof:** By Lemma 10, if a UMP test exists then $\tilde{f}_{\alpha, a}$ is also a UMP test. Therefore, it suffices to prove that $\tilde{f}_{\alpha, a}$ is not a level-α UMP test. To this end, we explicitly construct a test $f$ and prove that the rankings assigned value 1 are more cost-effective than that under $\tilde{f}_{\alpha, a}$.

Let $V_1, V_2, \ldots, V_m, V'_2 \in \mathcal{L}(A)$ denote $m + 1$ rankings defined as follows. For any $j \leq m$, let $V_j = [a_j \succ \text{others}]$, where alternatives in “others” are ranked w.r.t. the increasing order of their subscripts. In other words, $V_j$ is obtained from $V'_1$ by raising alternative $a_j$ to the top position. We let $V''_2 = [a_3 \succ a_1 \succ a_4 \succ a_2 \succ \text{others}]$.

We consider the following critical function $f$. For any $V \in \mathcal{L}_{a \succ \text{others}}$, we let $f(V) = 1$. For any $V_j$ with $j \neq 3$, let $f(V_j) = 1$. We then let $f(V_3) = f(V'_3) = \frac{1 + \varphi^m}{1 + \varphi}$. Let $\alpha$ denote the size of $f$ at $V_2$. That is, $\alpha = \text{Size}(f, V_2)$. Let $T = \pi V_2(\mathcal{L}_{a \succ \text{others}})$. It follows that

\[
\begin{align*}
\alpha - T &= \varphi^0 + \frac{1 + \varphi^m}{1 + \varphi} (\varphi^{KT(V_2, V_3)} + \varphi^{KT(V_2, V'_3)}) + \sum_{j=5}^{m} \varphi^{KT(V_2, V_j)} \\
&= 1 + \frac{1 + \varphi^m}{1 + \varphi} (\varphi^3 + \varphi^4) + \varphi^4 + \sum_{j=5}^{m} \varphi^{KT(V_2, V_j)} \\
&> 1 + \varphi^3 + \varphi^4 + \varphi^5
\end{align*}
\]

Figure 1: Kendall-Tau distance for some rankings over four alternatives.

For any $j, j^* \geq 2$ such that $j \neq j^*$, it is not hard to verify that $\text{KT}(V_j, V_{j^*}) = j + j^* - 2$. Moreover, $\text{KT}(V_3, V'_3) = 4$, $\text{KT}(V_4, V'_3) = 4$, and for any $j \geq 5$, we have $\text{KT}(V'_3, V_j) = j + 2$. Therefore, we have the following calculations of $\text{Size}(f, V_3)$, $\text{Size}(f, V'_3)$, and $\text{Size}(f, V_4)$ (see Figure 1 for distances between $V_2, V_3, V'_3, V_4$).

We note that $T = \pi V_2(\mathcal{L}_{a \succ \text{others}}) = \pi V_3(\mathcal{L}_{a \succ \text{others}}) = \pi V'_3(\mathcal{L}_{a \succ \text{others}}) = \pi V_4(\mathcal{L}_{a \succ \text{others}})$ due to symmetry.

\[
\begin{align*}
\text{Size}(f, V_3) - T &\leq 1 + \varphi^3 + \frac{1 + \varphi^m}{1 + \varphi} (\varphi^3 + \varphi^4 + \varphi^5 + \sum_{j=5}^{m} \varphi^{KT(V_3, V_j)}) \\
&= 1 + \varphi^3 + (m - 3)\varphi^5
\end{align*}
\]

\[
\begin{align*}
\text{Size}(f, V'_3) - T &\leq 1 + \varphi^4 + \frac{1 + \varphi^m}{1 + \varphi} (\varphi^3 + \varphi^4 + \varphi^5 + \sum_{j=5}^{m} \varphi^{KT(V'_3, V'_j)}) \\
&= 1 + 2\varphi^4 + (m - 4)\varphi^6
\end{align*}
\]

\[
\begin{align*}
\text{Size}(f, V_4) - T &\leq 1 + \varphi^4 + \frac{1 + \varphi^m}{1 + \varphi} (\varphi^3 + \varphi^4 + \varphi^5 + \sum_{j=5}^{m} \varphi^{KT(V_4, V'_j)}) \\
&= 1 + 2\varphi^4 + (m - 4)\varphi^7
\end{align*}
\]

For any other $h'_0 \in H_0$, we have $\text{Size}(f, h'_0) - T \leq m\varphi$. Because $\varphi < 1/m$, we have $\text{Size}(f) = \alpha$. Let $P$ denote a profile that is composed of $\{V_2, V_4, \ldots, V_m\} \cup \frac{1 + \varphi^m}{1 + \varphi} \{V_3, V'_3\}$. We next prove that $\text{Ratio}_{V_2, V_1}(P) > \text{Ratio}_{V_2, V_1}(T_{m-2})$. 

Let $Z_m = \prod_{i=1}^m \frac{1 - \varphi^m}{1 - \varphi}$ denote the Mallows normalization factor for $m$ alternatives. We have

\[
\text{Ratio}_{\nu_2, \nu_1}(T_{m-2}) = \frac{\nu_1(T_{m-2})}{\nu_2(T_{m-2})} = \frac{\varphi Z_{m-1}}{Z_{m-2} + \varphi^2(Z_{m-1} - Z_{m-2})} = \frac{\frac{\varphi Z_{m-1}}{Z_{m-2}}}{1 + \varphi^2(Z_{m-1} - Z_{m-2}) - 1} = \frac{\varphi + \varphi^2 + \cdots + \varphi^{m-1}}{1 + \varphi^3 + \varphi^4 + \cdots + \varphi^m} < \frac{1}{\varphi}
\]

\[
\text{Ratio}_{\nu_2, \nu_1}(P) = \frac{\varphi + \varphi^2 + \cdots + \varphi^{m-1} + \varphi^{m+2}}{1 + \varphi^3 + \varphi^4 + \cdots + \varphi^m} > \frac{1}{\varphi} = \text{Ratio}_{\nu_2, \nu_1}(T_{m-2})
\]

We note that $\text{Size}(\bar{f}_{\alpha, \nu_2}) = \alpha$. This means that $\text{Power}(\bar{f}_{\alpha, \nu_1}) = \nu_1(T_{m-1}) + \alpha \text{Ratio}_{T_2, T_1}(T_{m-2}) < \nu_1(T_{m-1}) + \alpha \text{Ratio}_{T_2, T_1}(P) = \text{Power}(f, V_1)$. This means that $\bar{f}_{\alpha, \nu_1}$ is not a level-$\alpha$ UMP. The theorem follows after Lemma 10.

**Theorem 5.** Let $\mathcal{M}_{M^{\alpha}}$ denote a Mallows’ model with $n = 1$ and any $m \geq 4$. There exists $\epsilon > 0$ such that for any $\varphi > 1 - \epsilon$ and any $\alpha$, $\bar{f}_{\alpha, \nu_1}$ is a UMP test for $H_0 = (\mathcal{L}(A) - H_1)$ vs. $H_1 = L_{\alpha, \text{others}}$.

**Proof:** We first verify that when $K_\alpha = m - 1$, $\bar{f}_{\alpha, \nu_1}$ is a UMP test. For any $h_1 \in H_1$, let $h_0^\alpha \in H_0$ denote the ranking that is obtained from $h_1$ by moving $a$ down for one position. It is not hard to check that for any $V \in \mathcal{L}(A)$, $\text{Ratio}_{h_0^\alpha, h_1}(V) \leq 1/\varphi$, and for all $V \in H_1$ we have $\text{Ratio}_{h_0^\alpha, h_1}(V) = 1/\varphi$. This means that for any level-$\alpha$ test for $H_0$ vs. $h_1$, the power cannot be more than $\alpha/\varphi$. We note that $\bar{f}_{\alpha, \nu_1}$ is a level-$\alpha$ test whose power is exactly $\alpha/\varphi$. This means that for all $h_1 \in H_1$, $\bar{f}_{\alpha, \nu_1}$ is a most powerful test for $H_0$ vs. $h_1$. Therefore, when $K_\alpha = m - 1$, $\bar{f}_{\alpha, \nu_1}$ is a UMP test.

For any $\alpha$ such that $K_\alpha \leq m - 2$, we will prove that for any $h_1 \in H_1$, $\bar{f}_{\alpha, \nu_1}$ is a most powerful level-$\alpha$ test for $H_0$ vs. $h_1$. This is done in the following steps. Step 1. Find a least favorable distribution $\Lambda_{h_1}^{\alpha}$ whose support is the set of all rankings where $a$ is ranked at the second position. Step 2. Verify that $\bar{f}_{\alpha, \nu_1}$ is the likelihood ratio test w.r.t. $\Lambda_{h_1}^{\alpha}$, and step 3. verify that the two conditions in Lemma 2 holds for $\Lambda_{h_1}^{\alpha}$.

**Step 1.** The main challenge is that in general there does not exist a uniformly least favorable distribution. For different $\alpha$ we define different $\Lambda_{h_1}^{\alpha}$ as follows. For any $\alpha$, we let $s_\alpha$ denote the smallest Borda score of the ranking $V$ such that $\bar{f}_{\alpha, \nu_1}(V) > 0$. We have that $s_\alpha \leq m - 2$. Let the support of $\Lambda_{h_1}^{\alpha}$ be $T_{m-2}$, which is the set of rankings where $a$ is ranked at the second position. We will solve the following system of linear equations to determine $\Lambda_{h_1}^{\alpha}$. For any $h_0^\alpha \in T_{m-2}$ there is a variable $x[h_0^\alpha, s_\alpha]$.

\[
\forall V \in T_{s_\alpha}, \sum_{h_0^\alpha \in T_{m-2}} \text{Ratio}_{h_0^\alpha, h_1}(V) \cdot x[h_0^\alpha, s_\alpha] = m \quad (LP_{h_1}^{\alpha})
\]

We note that as $\varphi \to 1$, $\text{Ratio}_{h_0^\alpha, h_1}(V) = \frac{\pi_{h_1}(V)}{\pi_{h_0^\alpha}(V)} = \varphi^{KT(h_0^\alpha, V)} - KT(h_1, V) \to 1$. Because there are $m$ variables and $m$ equations, as $\varphi \to 1$ the solution to $LP_{h_1}^{\alpha}$ converges to $\bar{\alpha}$. Therefore, there exists $\epsilon > 0$ such that for all $\varphi > 1 - \epsilon$, the linear systems $\{LP_{h_1}^{\alpha} : s \leq m - 1, h_1 \in H_1\}$ all have strictly positive solutions. Let $\{x^*[h_0^\alpha, s_\alpha] | V \in T_{s_\alpha}\}$ denote a solution to $LP_{h_1}^{\alpha}$. For any $h_0^\alpha \in T_{m-2}$, we let $\Lambda_{h_1}^{\alpha}(h_0^\alpha) = \frac{x^*[h_0^\alpha, s_\alpha] - 1}{\sum_{h_0^\alpha \in T_{m-2}} x^*[h_0^\alpha, s_\alpha] - 1}$.

**Step 2.** To simplify notation we let $\text{LR}_\alpha = \text{LR}_{\alpha, \Lambda_{h_1}^{\alpha}}$ denote the likelihood ratio test and let $\text{Ratio} = \text{Ratio}_{\Lambda_{h_1}^{\alpha}}$ denote the likelihood ratio function w.r.t. distribution $\Lambda_{h_1}^{\alpha}$ for $H_0$ vs. $h_1$. To prove $\text{LR}_\alpha = \bar{f}_{\alpha, \nu_1}$, we first prove that for any $V \in \mathcal{L}(A)$ where $a$ is not ranked at the bottom position, $\text{Ratio}(V) > \text{Ratio}(\text{Down}_{a}(V))$, where we recall that
\( \text{Down}_a^1(V) \) is the ranking obtained from \( V \) by moving \( a \) down for one position.

\[
\frac{\sum_{h_0^* \in T_{m-2}} \Lambda^{h_1}_\alpha(h_0^*) \cdot \pi_{h_0^*}(\text{Down}_a^1(V))}{\sum_{h_0^* \in T_{m-2}} \Lambda^{h_1}_\alpha(h_0^*) \cdot \pi_{h_0^*}(V)}
= \frac{\sum_{h_0^* \in T_{m-2}} \Lambda^{h_1}_\alpha(h_0^*) \cdot \varphi(K_T(h_0^*, \text{Down}_a^1(V)))}{\sum_{h_0^* \in T_{m-2}} \Lambda^{h_1}_\alpha(h_0^*) \cdot \varphi(K_T(h_0^*, V))}
> \frac{\sum_{h_0^* \in T_{m-2}} \Lambda^{h_1}_\alpha(h_0^*) \cdot \varphi(K_T(h_0^*, V) \cdot \varphi(K_T(V, \text{Down}_a^1(V)))}{\sum_{h_0^* \in T_{m-2}} \Lambda^{h_1}_\alpha(h_0^*) \cdot \varphi(K_T(h_0^*, V))}
= \varphi = \frac{\pi_{h_1}(\text{Down}_a^1(V))}{\pi_{h_1}(V)}
\]

The strict inequality holds because of (1) triangle inequality for Kendall-Tau distance, and (2) for any ranking \( V \) where the top-ranked alternative in \( h_0^* \) is ranked right below \( a \), we have \( KT(h_0^*, V) + KT(V, \text{Down}_a^1(V)) > KT(h_0^*, \text{Down}_a^1(V)) \), and (3) for all \( h_0^* \in T_{m-2} \), \( \Lambda^{h_1}_\alpha(h_0^*) > 0 \).

It follows from the strict inequality that

\[
\text{Ratio}(V) = \frac{\pi_{h_1}(V)}{\sum_{h_0^* \in T_{m-2}} \Lambda^{h_1}_\alpha(h_0^*) \cdot \pi_{h_0^*}(V)}
> \frac{\pi_{h_1}(\text{Down}_a^1(V))}{\sum_{h_0^* \in T_{m-2}} \Lambda^{h_1}_\alpha(h_0^*) \cdot \pi_{h_0^*}(\text{Down}_a^1(V))}
\]

Moreover, for any \( V, V' \in T_{s_a} \), we have \( \text{Ratio}(V) = \text{Ratio}(V') \) by verifying \( LP_{s_a}^{h_1} \). Therefore, for any \( V \in T_i \) with \( i < s_\alpha \), we can move up the position of \( a \) one by one until we reach the \( (m - s_\alpha) \)-th position. Let \( V^* \in T_{s_a} \) denote this ranking. It follows that \( \text{Ratio}(V) < \text{Ratio}(V^*) \). Similarly for any \( V' \in T_i \) with \( i > s_\alpha \), we have \( \text{Ratio}(V') > \text{Ratio}(V^*) \) for any \( V^* \in T_{s_a} \). This means that for any \( V \) where \( a \) is ranked above the \( (m - s_\alpha) \)-th position, we have \( LR_{s_\alpha}(V) = 1 \); for any \( V \) where \( a \) is ranked below the \( (m - s_\alpha) \)-th position, we have \( LR_{s_\alpha}(V) = 0 \); for any \( V \) where \( a \) is ranked at the \( (m - s_\alpha) \)-th position, we have that \( LR_{s_\alpha}(V) \) is the same and is between 0 and 1. It follows that \( LR_{s_\alpha} = f_{s_a,a} \).

**Step 3.** Due to the symmetry \( f_{s,a} \) among alternatives in \( A \setminus \{a\} \), for any \( i \leq m - 2 \) and any \( h_0, h_0' \in T_i \), we have \( \text{Size}(f_{s,a}, h_0^*) = \text{Size}(f_{s,a}, h_0'^*) \). Therefore, condition (i) in Lemma 2 is satisfied. Choose arbitrary \( h_0^m \in T_{m-2} \) and any \( h_0^* \in T_i \) denote the ranking obtained from \( h_0^m \) by moving \( a \) down for one position. To verify condition (ii) in Lemma 2, it suffices to prove that for any \( i \leq m - 3 \) and any \( K \in \mathbb{N} \), we have

\[
\pi_{h_0}^{m-2}(\{V : \text{Borda}_a(V) \geq K\}) \geq \pi_{h_0}^1(\{V : \text{Borda}_a(V) \geq K\})
\tag{2}
\]

We will prove a slightly stronger lemma.

**Lemma 8** Under Mallows’ model, for any \( m, \varphi, \) any \( W \in \mathcal{L}(A) \), any \( b, c \in A \) such that \( b \succ_W c \), and any \( K \), we have \( \pi_W(\{V : \text{Borda}_a(V) \geq K\}) \geq \pi_W(\{V : \text{Borda}_c(V) \geq K\}) \).

**Proof:** The proof is similar to the proof of Lemma 5. It suffices to prove the lemma for the case where \( b \) and \( c \) are adjacent in \( W \). Let \( \mathcal{P} = \{V \in \mathcal{L}(A) : \text{Borda}_a(V) \geq K\} \) and \( \mathcal{P}' = \{V \in \mathcal{L}(A) : \text{Borda}_c(V) \geq K\} \). It follows that \( \mathcal{P} \cap \mathcal{P}' \) is the set of rankings where both \( b \) and \( c \) are ranked within top \( m - K \) positions; \( \mathcal{P} - \mathcal{P}' \) is the set of rankings where \( b \) is ranked within top \( m - K \) positions but \( c \) is not; and \( \mathcal{P}' - \mathcal{P} \) is the set of rankings where \( c \) is ranked within top \( m - K \) positions but \( b \) is not. We let \( \mathcal{M} \) be a permutation that switches \( b \) and \( c \). It is not hard to check that \( \mathcal{M} \) is a bijection between \( \mathcal{P} - \mathcal{P}' \) and \( \mathcal{P}' - \mathcal{P} \), and because \( b \) and \( c \) are adjacent in \( W \), for any \( V \in \mathcal{P} \), we have \( KT(M(V), W) = KT(V, W) + 1 \), which means that \( \pi_W(V) = \pi(M(V))/\varphi \). Therefore, we have
Lemma 7. For any model $\mathcal{M}_X$ and any $t \in \mathbb{N}$, suppose $\Lambda$ is a uniformly least favorable distribution for composite vs. simple test $(H_0, h_1)$ under $\mathcal{M}_X$. Then $\text{Ext}(\Lambda, h_1, t)$ is a uniformly least favorable distribution for $\text{Ext}(H_0, h_1, t)$ vs. $\bar{h}_1$ in $(\mathcal{M}_X)^t$. 

Lemma 6. For any $\mathcal{M}_X$ and $\mathcal{M}_Y$, suppose $\Lambda_X$ is a least favorable distribution for composite vs. simple test $(H_0, X)$ under $\mathcal{M}_X$. Given $y_1 \in \Theta_Y$, let $\Lambda^*$ be the distribution over $H_0, X \times \Theta_Y$ where for all $x \in H_0, X$, $\Lambda^*(x, y_1) = \Lambda_X(x)$. Then $\Lambda^*$ is a least favorable distribution for $H_0, X \times \Theta_Y$ vs. $(x_1, y_1)$ under $\mathcal{M}_X \otimes \mathcal{M}_Y$.

Proof: Let $x_0^K, \ldots, x_0^K \in \Theta_X$ denote the support of $\Lambda_X$. The theorem is proved by applying Lemma 2. For any $0 < \alpha < 1$ and any $P = (P_X, P_Y) \in S_X \times S_Y$, we have the following calculation. In this proof $\text{Ratio}$ stands for $\text{Ratio}_{\Lambda^*, (x_1, y_1)}$ and $\text{LR}_\alpha$ stands for $\text{LR}_{\alpha, \Lambda^*, (x_1, y_1)}$.

\[
\text{Ratio}(P_X, P_Y) = \frac{\sum_{k=1}^{K} \Lambda^*(x_0^k, y_1) \pi(x_0^k, y_1)}{\sum_{k=1}^{K} \pi(x_0^k, y_1)} = \frac{\sum_{k=1}^{K} \Lambda^*(x_0^k, y_1) \pi(x_0^k)}{\sum_{k=1}^{K} \pi(x_0^k)} = \text{Ratio}_{\Lambda^*, (x_1, y_1)}(P_X)
\]

It follows that for any pair of samples $(P_X, P_Y), (P_X', P_Y') \in S_X \times S_Y$, $\text{Ratio}(P_X, P_Y) \geq \text{Ratio}(P_X', P_Y')$ if and only if $\text{Ratio}_{\Lambda^*, (x_1, y_1)}(P_X) \geq \text{Ratio}_{\Lambda^*, (x_1, y_1)}(P_X')$. This means that for any $(P_X, P_Y)$, $\text{LR}_\alpha(P_X, P_Y) = \text{LR}_\alpha, \Lambda^*, (x_1, y_1)(P_X)$. Therefore, for any $x_0 \in H_0, X$, we have

\[
\text{Size}(\text{LR}_\alpha, (x_0, y_1)) = \sum_{(P_X, P_Y) \in S_X \times S_Y} \pi(x_0, P_X) \pi(y_1, P_Y) \text{LR}_\alpha(P_X, P_Y) = \sum_{(P_X, P_Y) \in S_X \times S_Y} \pi(x_0, P_X) \pi(y_1, P_Y) \text{LR}_{\alpha, \Lambda^*, (x_1, y_1)}(P_X)
\]

Therefore, by Lemma 2, for any $(x_0^k, y_1) \in \text{Spt}(\Lambda^*)$, we have $\text{Size}(\text{LR}_\alpha, (x_0, y_1)) = \text{Size}(\text{LR}_\alpha, \Lambda^*, X_1, x_0) = \alpha$ because $x_0^k \in \text{Spt}(\Lambda)$; for any $(x_0, y) \in H_0, X \times \Theta_Y$, we have $\text{Size}(\text{LR}_\alpha, (x_0, y)) = \text{Size}(\text{LR}_\alpha, \Lambda^*, X_1, x_0) \leq \alpha$. This means that the two conditions in Lemma 2 are satisfied, which proves the theorem.

Lemma 7. For any model $\mathcal{M}_X$ and any $t \in \mathbb{N}$, suppose $\Lambda$ is a uniformly least favorable distribution for composite vs. simple test $(H_0, h_1)$ under $\mathcal{M}_X$. Then $\text{Ext}(\Lambda, h_1, t)$ is a uniformly least favorable distribution for $\text{Ext}(H_0, h_1, t)$ vs. $\bar{h}_1$ in $(\mathcal{M}_X)^t$. 

This proves the lemma.
Claim 1 For any \( k_\alpha \) and any \( \bar{x} \in S^t \), \( \sum_{j=1}^t \text{Ratio}_{\Lambda, h_1}^{-1}(x_j) = t \cdot \text{Ratio}_{\Lambda}^{-1}(\bar{x}) \).

Proof: we have \( \text{Ratio}_{\Lambda}^{-1}(\bar{x}) = \frac{1}{t} \sum_{j=1}^t \sum_{h_0 \in H_0} \Lambda(h_0) \cdot \pi_{h_0, [\pi]\_1\_j}(\bar{x})}{\pi_{\Lambda}(\bar{x})} = \frac{1}{t} \sum_{j=1}^t \text{Ratio}_{\Lambda, h_1}^{-1}(x_j) \)

The next lemma proves the following: For any \( \bar{x} \in H_0^\ast \) and any \( j \leq t \), suppose the \( j \)-th component is not in \( \text{Spt}(\Lambda) \cup \{ h_1 \} \). If we fix all components except \( j \)-th in \( \bar{x} \) and change the \( j \)-th component to \( h_0^\ast \in \text{Spt}(\Lambda) \), then the size of \( \text{LR}_\alpha \) will increase. If we further change the \( j \)-th component to \( h_1 \), then the size of \( \text{LR}_\alpha \) will further increase.

Lemma 9 For any \( 0 \leq \alpha \leq 1 \), any \( j \leq t \), any \( \bar{x}, j \in \Theta^{-1} \), any \( h_0 \in H_0 \), and any \( h_0^\ast \in \text{Spt}(\Lambda) \), we have \( \text{Size}(\text{LR}_\alpha, (h_0, \bar{x})) \leq \text{Size}(\text{LR}_\alpha, (h_0^\ast, \bar{x})) \leq \text{Size}(\text{LR}_\alpha, (h_1, \bar{x})) \).

Proof: For any \( \bar{x}, j \in \Theta^{-1} \), we have
\[
\text{Size}(\text{LR}_\alpha, (h_0, \bar{x})) = \pi_{(h_0, \bar{x})}(\{ \bar{x} \in S^t : \text{Ratio}(\bar{x}) > k_\alpha^\ast \}) + \gamma_\alpha^\ast \pi_{(h_0, \bar{x})}(\{ \bar{x} \in S^t : \text{Ratio}(\bar{x}) = k_\alpha^\ast \})
\]
For any \( \bar{x} \), we let \( \text{Sum}(\bar{x}) = \sum_{i=t}^t \text{Ratio}_{\Lambda, h_1}^{-1}(x_i) \) and for any \( j \leq t \), we let \( \text{Sum}(\bar{x}, j) = \sum_{i \neq j} \text{Ratio}_{\Lambda, h_1}^{-1}(x_i) \). By Claim 1, we have
\[
\pi_{(h_0, \bar{x})}(\{ \bar{x} \in S^t : \text{Ratio}(\bar{x}) > k_\alpha^\ast \}) = \pi_{(h_0, \bar{x})}(\{ \bar{x} \in S^t : \text{Sum}(\bar{x}) < t/k_\alpha^\ast \}) = \pi_{(h_0, \bar{x})}(\{ \bar{x} \in S^t : \text{Sum}(\bar{x}) + \text{Ratio}_{\Lambda, h_1}^{-1}(x_j) < t/k_\alpha^\ast \})
\]
\[
= \int_0^{t/k_\alpha^\ast} \sum_{x_j \in S^t : \text{Sum}(\bar{x}, j) = p} \frac{1}{t/k_\alpha^\ast} \pi_{(h_0, \bar{x})}(\bar{x}) dp
\]
\[
= \int_0^{t/k_\alpha^\ast} \pi_{(h_0, \bar{x})}(\{ \bar{x} \in S^t : \text{Sum}(\bar{x}, j) = p \}) \cdot \pi_{h_0}(\{ x_j : \text{Ratio}_{\Lambda, h_1}^{-1}(x_j) < t/k_\alpha^\ast - p \}) dp
\]
where \( Q(\bar{x}, p) = \pi_{(h_0, \bar{x})}(\{ \bar{x} \in S^t : \text{Sum}(\bar{x}, j) = p \}) \). Given \( p \) and \( \gamma_\alpha^\ast \), let \( \alpha' \) denote the size of the likelihood ratio test \( \text{LR}_{\alpha', \Lambda, h_1} \), where the threshold \( k_{\alpha'} \) is \( 1/(t/k_\alpha^\ast - p) \) and \( \gamma_{\alpha'} = \gamma_\alpha^\ast \). We have
\[
\text{Size}(\text{LR}_\alpha, (h_0, \bar{x})) = \int_0^{t/k_\alpha^\ast} Q(\bar{x}, p) \cdot \text{Size}(\text{LR}_{\alpha', \Lambda, h_1}, h_0) dp
\]
(3)
We note that in Equation (3), \( \alpha' \) is a function of \( t, p, k_{\alpha'}^\ast \), and \( \gamma_{\alpha'}^\ast \). Because \( \Lambda \) is a uniformly least favorable distribution, it follows from Lemma 2 that for any \( h_0^\ast \in \text{Spt}(\Lambda) \) and any \( h_0 \in (H_0 - \text{Spt}(\Lambda)) \), we have
\[
\text{Size}(\text{LR}_{\alpha', \Lambda, h_1}, h_0) \leq \alpha' \leq \text{Size}(\text{LR}_{\alpha', \Lambda, h_1}, h_0^\ast)
\]
Then by Equation (3), for any \( h_0 \in (H_0 - \text{Spt}(\Lambda)) \) and any \( h_0^* \in \text{Spt}(\Lambda) \), we have

\[
\text{Size}(\text{LR}_\alpha, (h_0, \bar{z}_{-j})) = \int_0^{t/k_\alpha^*} Q(\bar{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha' \cdot \Lambda, h_1}, h_0) dp
\]

\[
\leq \int_0^{t/k_\alpha^*} Q(\bar{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha' \cdot \Lambda, h_1}, h_0^*) dp
= \text{Size}(\text{LR}_\alpha, (h_0^*, \bar{z}_{-j}))
\]

To prove the last inequality in the lemma, we prove a claim that holds for any least favorable distribution and the corresponding likelihood ratio test. The \( \text{Size}(\cdot) \) function in the claim is extended to \( h_1 \in H_1 \) in the natural way.

**Claim 2** For any model, any composite vs. simple test \((H_0 \text{ vs. } h_1)\), suppose \( \Lambda \) is a level-\( \eta \) least favorable distribution. Then we have \( \text{Size}(\text{LR}_\eta, h_1) \geq \eta = \text{Size}(\text{LR}_\eta, h_0^\Lambda) \).

**Proof:** For the sake of contradiction suppose this is not true, that is, for any \( h_0^* \in \text{Spt}(\Lambda) \) we have \( \text{Size}(\text{LR}_\eta, h_1) < \eta = \text{Size}(\text{LR}_\eta, h_0^\Lambda) \). It follows that \( k_\eta < 1 \), otherwise we have

\[
\text{Size}(\text{LR}_\eta, h_1) = \sum_{P \in \mathcal{S}: \text{Ratio}(P) > k_\eta} \pi_{h_1}(P) + \gamma_\eta \sum_{P \in \mathcal{S}: \text{Ratio}(P) = k_\eta} \pi_{h_1}(P)
\]

\[
\geq \sum_{P \in \mathcal{S}: \text{Ratio}(P) > k_\eta} \pi_\Lambda(P) \cdot k_\eta + \gamma_\eta \sum_{P \in \mathcal{S}: \text{Ratio}(P) = k_\eta} \pi_\Lambda(P) \cdot k_\eta
\]

\[
> \sum_{P \in \mathcal{S}: \text{Ratio}(P) > k_\eta} \pi_\Lambda(P) + \gamma_\eta \sum_{P \in \mathcal{S}: \text{Ratio}(P) = k_\eta} \pi_\Lambda(P) = \eta,
\]

which is a contradiction. Therefore, we have

\[1 = \text{Size}(\text{LR}_\eta, h_1) + \sum_{P \in \mathcal{S}: \text{Ratio}(P) < k_\eta} \pi_{h_1}(P) + (1 - \gamma_\eta) \sum_{P \in \mathcal{S}: \text{Ratio}(P) = k_\eta} \pi_{h_1}(P)
\]

\[
< \eta + \sum_{P \in \mathcal{S}: \text{Ratio}(P) < k_\eta} \pi_\Lambda(P) \cdot k_\eta
\]

\[
+ (1 - \gamma_\eta) \sum_{P \in \mathcal{S}: \text{Ratio}(P) = k_\eta} \pi_\Lambda(P) \cdot k_\eta
\]

\[
\leq \eta + k_\eta(1 - \text{Size}(\text{LR}_\eta, h_0^\Lambda)) \leq 1,
\]

which is a contradiction. \( \square \)

Applying Claim 2 to \( \text{LR}_{\alpha' \cdot \Lambda, h_1} \), we have

\[
\text{Size}(\text{LR}_\alpha, (h_0^*, \bar{z}_{-j})) = \int_0^{t/k_\alpha^*} Q(\bar{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha' \cdot \Lambda, h_1}, h_0^*) dp
\]

\[
\leq \int_0^{t/k_\alpha^*} Q(\bar{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha' \cdot \Lambda, h_1}, h_1) dp
= \text{Size}(\text{LR}_\alpha, (h_1, \bar{z}_{-j}))
\]

\[4\text{We recall that } h_0^\Lambda \text{ is the combined } H_0 \text{ by } \Lambda.\]
This finishes the proof of Lemma 9.

It follows from Lemma 9 that for any \( j \leq t \) and any \( h^*_0 \in \text{Spt}(\Lambda) \), we have that \( \text{Size}(LR_\alpha, (h^*_0, [\bar{h}_1]_{-j})) \) is the same. Due to symmetry, for any \( \bar{h}_0 \in H_0^t \), \( \text{Size}(LR_\alpha, h^*_0) \) is the same and is therefore equivalent to \( \alpha \). This verifies condition (i) in Lemma 2.

Condition (ii) in Lemma 2 is verified by recursively applying Lemma 9. Given any \( \bar{h}_0 \in H_0^t - \text{Spt}(\Lambda^*) \), there must exist \( j \leq t \) such that \( [\bar{h}_0]_j \neq h_1 \). We then change \( [\bar{h}_0]_j \) to an arbitrary \( h^*_0 \in \text{Spt}(\Lambda) \), then change the other components of \( \bar{h}_0 \) to \( h_1 \) one by one. Each time we make the change the size of \( LR_\alpha \) does not decrease according to Lemma 9. At the end of the process we obtain \( (h^*_0, [\bar{h}_1]_j) \in \text{Spt}(\Lambda^*) \), at which the size of \( LR_\alpha \) is \( \alpha \). The theorem follows after applying Lemma 2.

We now define a test \( \tilde{f}_{\alpha,a} \) for \( H_0 = (\mathcal{L}(\mathcal{A}) - H_1) \) vs. \( H_1 = L_{a>\text{others}} \), and prove that if a UMP test exists, then \( \tilde{f}_{\alpha,a} \) must also be a UMP test. For any \( V \in \mathcal{L}(\mathcal{A}) \) and any alternative \( a \in \mathcal{A} \), we let \( \text{Borda}_a(V) \) denote the Borda score of \( a \) in \( V \). That is, \( \text{Borda}_a(V) \) is the number of alternatives that are ranked below \( a \) in \( V \). For any \( V \in \mathcal{L}(\mathcal{A}) \), we let

\[
\tilde{f}_{\alpha,a}(V) = \begin{cases} 
1 & \text{if } \text{Borda}_a(V) > K_\alpha \\
0 & \text{if } \text{Borda}_a(V) < K_\alpha \\
\Gamma_\alpha & \text{if } \text{Borda}_a(V) = K_\alpha
\end{cases}
\]

\( \tilde{f}_{\alpha,a} \) calculates the Borda score of \( a \) in the input profile, and if it is larger than a threshold \( K_\alpha \), then \( H_0 \) is rejected. It is not hard to see that \( f_{\alpha,a} \) equals to \( f_{\alpha,a} \) with a possibly different level \( \alpha' \) (defined in Theorem 3).

**Lemma 10** If there exists a level-\( \alpha \) UMP test for \( H_0 = (\mathcal{L}(\mathcal{A}) - H_1) \) vs. \( H_1 = L_{a>\text{others}} \), then \( \tilde{f}_{\alpha,a} \) is also a level-\( \alpha \) UMP test.

**Proof:** Let \( f_\alpha \) denote a level-\( \alpha \) UMP test. For any permutation \( M \) over \( \mathcal{A} - \{a\} \), we let \( M(f_\alpha) \) denote the test such that for any \( V \in \mathcal{L}(\mathcal{A}) \), \( M(f_\alpha)(V) = f_\alpha(M(V)) \). Because the Kendall-Tau distance is invariant to permutations, we have that for any \( h_0 \in H_0 \), \( \text{Size}(f_\alpha, h_0) = \text{Size}(M(f_\alpha), M(h_0)) \), and for any \( h_1 \in H_1 \), \( \text{Power}(f_\alpha, h_1) = \text{Power}(M(f_\alpha), M(h_1)) \). Therefore \( \text{Size}(M(f_\alpha)) = \alpha \). Also because the multi-set of \( \{\text{Power}(f_\alpha, h_1) : h_1 \in H_1\} \) is the same as the multi-set \( \{\text{Power}(M(f_\alpha), h_1) : h_1 \in H_1\} \), for all \( h_1 \in H_1 \), we must have \( \text{Power}(f_\alpha, h_1) = \text{Power}(M(f_\alpha), h_1) \), otherwise there exists \( h_1 \in H_1 \) such that \( \text{Power}(f_\alpha, h_1) < \text{Power}(M(f_\alpha), h_1) \), which contradicts the assumption that \( f_\alpha \) is UMP.

It follows that for any permutation \( M \) over \( \mathcal{A} - \{a\} \), \( M(f_\alpha) \) is also UMP. Therefore, \( \tilde{f}_\alpha = \frac{1}{(m-1)!} \sum M(f_\alpha) \) is also UMP. We note that for any \( V, V' \) where \( a \) has the same Borda score, there exists a permutation \( M \) over \( \mathcal{A} - \{a\} \) so that \( M(V) = V' \). This means that \( \tilde{f}_\alpha(V) = \tilde{f}_\alpha(V') \).

We now prove that \( f_\alpha \) must be \( \tilde{f}_{\alpha,a} \) as in the statement of the Lemma. More precisely, we will prove that for any \( V, V' \) such that \( \text{Borda}_a(V) > \text{Borda}_a(V') \), if \( \tilde{f}_\alpha(V') > 0 \) then \( \tilde{f}_\alpha(V) = 1 \). Suppose for the sake of contradiction that this is not true, and there exist \( V, V' \) such that \( s_1 = \text{Borda}_a(V) > \text{Borda}_a(V') = s_2, \tilde{f}_\alpha(V') > 0, \) and \( \tilde{f}_\alpha(V) < 1 \). For any \( s \leq m - 1 \), we let \( T_s \) denote the set of rankings where the Borda score of \( a \) is \( s \). That is, \( T_s = \{V \in \mathcal{L}(\mathcal{A}) : \text{Borda}_a(V) = s \} \). We will prove that for any \( s_1 > s_2, T_{s_1} \) as a whole is more “cost effective” than \( T_{s_2} \) as a whole for any \( h_0 \in H_0 \) against any \( h_1 \in H_1 \). More precisely, we will prove that \( \text{Ratio}_{h_0, h_1}(T_{s_1}) > \text{Ratio}_{h_0, h_1}(T_{s_2}) \).

For any \( s \leq m - 2 \) and any \( h_0 \in T_s \), let \( h_1 \) denote the ranking in \( T_{s-1} = H_1 \) that is obtained from \( \theta \) by raising \( a \) to the top position. For any \( V_{s_1} \in T_{s_1} \), we let \( \text{Down}_{s_1, s_2}(V_{s_1}) \in T_{s_2} \) denote the ranking that is obtained from \( V_{s_1} \) by
moving a down for \( s_1 - s_2 \) positions, that is, from the \((m - s_1)\)-th position to the \((m - s_2)\)-th position. We have

\[
\frac{\pi_{h_0}(T_{s_2})}{\pi_{h_0}(T_{s_1})} = \frac{\sum_{V \in T_{s_2}} \pi_{h_0}(V)}{\sum_{V \in T_{s_1}} \pi_{h_0}(V)} = \frac{\sum_{V \in T_{s_1}} \pi_{h_0}(\text{Down}^a_{s_1-s_2}(V))}{\sum_{V \in T_{s_2}} \pi_{h_0}(V)}
\]

\[
= \frac{\sum_{V \in T_{s_1}} \phi_{\text{KT}(h_0,\text{Down}^a_{s_1-s_2}(V))}}{\sum_{V \in T_{s_2}} \phi_{\text{KT}(h_0,V)}}
\]

\[
> \frac{\sum_{V \in T_{s_1}} \phi_{\text{KT}(h_0,V)} \cdot \phi_{\text{KT}(V,\text{Down}^a_{s_1-s_2}(V))}}{\sum_{V \in T_{s_2}} \phi_{\text{KT}(h_0,V)}}
\]

\[
= \phi^{s_1-s_2} = \frac{\pi_{h_1}(T_{s_2})}{\pi_{h_1}(T_{s_1})}
\]

The inequality is due to triangle inequality for Kendall-Tau distance. It is strict because for any \( V \in T_{s_1} \), where the top-ranked alternative in \( h_0 \) is ranked between the \((m - s_1)\)-th and \((m - s_2)\)-th position, \( \text{KT}(h_0,\text{Down}^a_{s_1-s_2}(V)) < \text{KT}(h_0,V) + \text{KT}(V,\text{Down}^a_{s_1-s_2}(V)) \). Therefore, \( \frac{\pi_{h_0}(T_{s_2})}{\pi_{h_0}(T_{s_1})} > \frac{\pi_{h_1}(T_{s_2})}{\pi_{h_1}(T_{s_1})} \), which means that \( \text{Ratio}_{h_0,h_1}(T_{s_1}) = \frac{\pi_{h_1}(T_{s_1})}{\pi_{h_0}(T_{s_1})} > \frac{\pi_{h_1}(T_{s_2})}{\pi_{h_0}(T_{s_2})} = \text{Ratio}_{h_0,h_1}(T_{s_2}) \).

Therefore, we can find sufficiently small \( \epsilon, \delta > 0 \), and replace \( \epsilon T_{s_2} \) by \( \delta T_{s_1} \), without changing the size. This will increase the power of \( \hat{f}_\alpha \) because \( T_{s_1} \) is strictly more cost effective than \( T_{s_2} \), which contradicts the assumption that \( \hat{f}_\alpha \) is a UMP test. Therefore, \( \hat{f}_\alpha = \hat{f}_{\alpha,a} \), which proves the lemma. \( \square \)