Supplement to the article: Averaging of Decomposable Graphs by Dynamic Programming and Sampling

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A FAST TRANSFORM FOR DISJOINT PAIRS

Let α be a function that associates each pair of disjoint subsets of $V = \{1, \ldots, n\}$ with a real number. Define the function $\hat{\alpha}$ by letting

$$\hat{\alpha}(S, R) = \sum_{S \subseteq C \subseteq S \cup R} \alpha(C, R \setminus C)$$

for all disjoint subsets S and R of V. Furthermore, define $\alpha_0 = \alpha$ and for $i = 1, \ldots, n$, recursively

$$\alpha_i(S, R) = \alpha_{i-1}(S, R) + [i \in R] \cdot \alpha_{i-1}(S \cup \{i\}, R \setminus \{i\}) .$$

This recurrence gives us a way to compute the transform:

Lemma 8. *It holds that $\alpha_n = \hat{\alpha}$.*

Proof. For a subset $X \subseteq V$ and element $i \in V$, we write X_i for the set $X \cap \{1, \ldots, i\}$ and X^i for the set $X \cap \{i + 1, \ldots, n\} = X \setminus X_i$.

We will show by induction on i that

$$\alpha_i(S, R) = \sum_{S_i \subseteq C \subseteq (S \cup R)_i} \alpha(C \cup S^i, (R_i \setminus C) \cup R^i) . \quad (5)$$

This clearly holds for $i = 0$, as then the only term in the sum is $\alpha(\varnothing \cup S^0, (R_0 \setminus \varnothing) \cup R^0) = \alpha(S, R) = \alpha_0(S, R)$.

Suppose then that $i > 0$. Consider first the case that $i \notin R$. Then, by the definition and the induction hypothesis,

$$\alpha_i(S, R) = \sum_{S_{i-1} \subseteq C \subseteq (S \cup R)_{i-1}} \alpha(C \cup S^{i-1}, (R_{i-1} \setminus C) \cup R^{i-1}) .$$

Writing C' for the set $C \cup (S \cap \{i\})$ we now obtain

$$\alpha_i(S, R) = \sum_{S_i \subseteq C' \subseteq (S \cup R)_i} \alpha(C' \cup S^i, (R_i \setminus C') \cup R^i) ,$$

which matches the induction hypothesis (5).

Consider then the case that $i \in R$. Observe that $i \notin S$, since S is disjoint from R. As above, expand $\alpha_{i-1}(S, R)$ using the induction hypothesis into

$$\sum_{S_{i-1} \subseteq C \subseteq (S \cup R)_{i-1}} \alpha(C \cup S^{i-1}, (R_{i-1} \setminus C) \cup R^{i-1}) ,$$

which equals

$$\sum_{\substack{S_i \subseteq C' \subseteq (S \cup R)_i \\ i \notin C'}} \alpha(C' \cup S^i, (R_i \setminus C') \cup R^i) .$$

Likewise, expand $\alpha_{i-1}(S \cup \{i\}, R \setminus \{i\})$ using the induction hypothesis into

$$\sum_{S_{i-1} \subseteq C \subseteq (S \cup R)_{i-1}} \alpha(C' \cup S^{i-1}, (R_{i-1} \setminus C') \cup (R \setminus \{i\})^{i-1}) ,$$

where we write C' for $C \cup \{i\}$. Observe that this sum equals

$$\sum_{\substack{S_i \subseteq C' \subseteq (S \cup R)_i \\ i \in C'}} \alpha(C' \cup S^i, (R_i \setminus C') \cup R^i) ,$$

because $i \notin S$ and $i \in R$. Adding up the obtained two sums over C' yields

$$\alpha_i(S, R) = \sum_{S_i \subseteq C' \subseteq (S \cup R)_i} \alpha(C' \cup S^i, (R_i \setminus C') \cup R^i) ,$$

which matches the induction hypothesis (5). □

B PROOF OF LEMMA 4

In order to prove Lemma 4, we first prove the following lemma:

Lemma 9. *Backtracking starting from $g(C, U)$ makes at most $|U|$ recursive nonterminating visits to g (including the visit to $g(C, U)$).*

Proof. We show the claim by induction over $|U|$. The case $|U| = 0$ is trivial as it terminates. Suppose that $|U| \geq 1$ and the claim holds for smaller U. The visit to $g(C,U)$ is followed by recursive visits to (i) $g(C, U \setminus R)$ and (ii) $h(C,R)$. By the induction assumption (i) amounts to at most $|U|-|R|$ recursive nonterminating visits to g. Visit (ii) is followed by a visit to f succeeded by a visit to $g(C', R \setminus C')$ for some C' with $R \cap C' \neq \varnothing$. Thus, by the induction assumption, (ii) amounts to at most $|R|-1$ recursive nonterminating visits to g. The total, including the visit to $g(C,U)$, is thus at most $(|U|-|R|)+(|R|-1)+1=|U|$. $\square$

Now we can prove Lemma 4:

Proof of Lemma 4. Observe that the first two visits are to $f(\varnothing, V)$ and to $g(C,U)$ where $|U| \leq |V|-1$. By Lemma 9, there are thus at most $n-1$ nonterminating visits to g. Also note that a visit to h is always from a nonterminating visit to g and a visit to f always from a visit to h (except the first visit). The result follows. $\square$

C PROOF OF LEMMA 5

In order to prove Lemma 5, we first prove the following lemma:

Lemma 10. *Consider backtracking from $g(C,U)$ onwards. Let $\{(C_1,U_1),(C_2,U_2),\ldots,(C_d,U_d)\}$ be the set pairs of the recursive nonterminating visits to g, including (C,U). Then there exists an ordering of the d set pairs such that*

$$|C_i|+|U_i| \leq |C|+|U|-i+1 \textit{ for all } i=1,\ldots,d.$$

Proof. We show the claim by induction over $|U|$. The case $|U|=0$ is trivial as there are no recursive visits. Suppose that $|U| \geq 1$ and the claim holds for smaller U. The visit to $g(C,U)$ is followed by recursive visits to (i) $g(C, U \setminus R)$ and (ii) $h(C,R)$.

First, let $(C_1,U_1)=(C,U)$. Clearly then the claim holds for $i=1$.

Let then $\{(C_2,U_2),\ldots,(C_{d'+1},U_{d'+1})\}$ be the d' set pairs of g visited in branch (i). By the induction assumption and the fact that $U \cap R \neq \varnothing$, there exists an ordering over these set pairs such that for all $i=2,\ldots,d'+1$,

$$\begin{aligned}|C_i|+|U_i| &\leq |C|+|U \setminus R|-(i-1)+1\\ &\leq |C|+|U|-i+1.\end{aligned}$$

Thus the claim holds for $i=2,\ldots,d'+1$.

Finally, branch (ii) makes, via h and f, a recursive visit to $g(C', R \setminus C')$ for some C' with $R \cap C' \neq \varnothing$. Let $(C_{d'+2},U_{d'+2}),\ldots,(C_{d'+d''+1},U_{d'+d''+1})$ be the d'' set pairs of g visited in branch (ii). By the induction assumption, there exists an ordering over these set pairs such that for all $i=d'+2,\ldots,d'+d''+1$,

$$\begin{aligned}|C_i|+|U_i| &\leq |C'|+|R \setminus C'|-(i-d'-1)+1\\ &\leq (|C|+|R|-1)-i+(|U|-|R|)+2\\ &= |C|+|U|-i+1.\end{aligned}$$

The second inequality uses the fact that $C' \cup R = S \cup R \subset C \cup R$, where S is selected during the visit to h, and the fact that by Lemma 9, $d' \leq |U \setminus R| = |U|-|R|$. As $d'+d''+1=d$, the claim thus holds for $i=d'+1,\ldots,d$, which completes the proof. $\square$

Now we can prove Lemma 5:

Proof of Lemma 5. For g the claim directly follows by applying Lemma 10. As a visit to $h(C,R)$ is always from a nonterminating visit to $g(C,U)$ with some U such that $C \cup R \subseteq C \cup U$, the claim follows for h. Finally, except the first visit, any other visit to $f(S,R)$ is always from a visit to $h(C,R)$ with some C such that $|S \cup R| \leq |C \cup R|-1$. Thus we can index the remaining set pairs from 2 to d_f such that the claim holds for them. Then (S_1,R_1) must be set to $(\varnothing, V)$ so that $|S_1|+|R_1| \leq n$ and the claim follows also for $i=1$. $\square$