A FAST TRANSFORM FOR DISJOINT PAIRS

Let $\alpha$ be a function that associates each pair of disjoint subsets of $V = \{1, \ldots, n\}$ with a real number. Define the function $\hat{\alpha}$ by letting

$$\hat{\alpha}(S, R) = \sum_{S \subseteq C \subseteq S \cup R} \alpha(C, R \setminus C)$$

for all disjoint subsets $S$ and $R$ of $V$. Furthermore, define $\alpha_0 = \alpha$ and for $i = 1, \ldots, n$, recursively

$$\alpha_i(S, R) = \alpha_{i-1}(S, R) + [i \in R] \cdot \alpha_{i-1}(S \cup \{i\}, R \setminus \{i\}).$$

This recurrence gives us a way to compute the transform:

**Lemma 8.** It holds that $\alpha_n = \hat{\alpha}$.

**Proof.** For a subset $X \subseteq V$ and element $i \in V$, we write $X_i$ for the set $X \cap \{1, \ldots, i\}$ and $X^i$ for the set $X \cap \{i+1, \ldots, n\} = X \setminus X_i$. We will show by induction on $i$ that

$$\alpha_i(S, R) = \sum_{S \subseteq C \subseteq (S \cup R)_i} \alpha(C \cup S^i, (R_i \setminus C) \cup R^i). \quad (5)$$

This clearly holds for $i = 0$, as then the only term in the sum is $\alpha(\emptyset \cup S^0_i, (R_0 \setminus \emptyset) \cup R^0) = \alpha(S, R) = \alpha_0(S, R)$. Suppose then that $i > 0$. Consider first the case that $i \notin R$. Then, by the definition and the induction hypothesis,

$$\alpha_i(S, R) = \sum_{S \subseteq C \subseteq (S \cup R)_{i-1}} \alpha(C \cup S^{i-1}, (R_{i-1} \setminus C) \cup R^{i-1}).$$

Writing $C'$ for the set $C \cup (S \cap \{i\})$ we now obtain

$$\alpha_i(S, R) = \sum_{S \subseteq C' \subseteq (S \cup R)_i} \alpha(C' \cup S^i, (R_i \setminus C') \cup R^i),$$

which matches the induction hypothesis (5).

Consider then the case that $i \in R$. Observe that $i \notin S$, since $S$ is disjoint from $R$. As above, expand $\alpha_{i-1}(S, R)$ using the induction hypothesis into

$$\sum_{S \subseteq C \subseteq (S \cup R)_{i-1}} \alpha(C \cup S^{i-1}, (R_{i-1} \setminus C) \cup R^{i-1}),$$

which equals

$$\sum_{S \subseteq C' \subseteq (S \cup R)_{i-1}} \alpha(C' \cup S^i, (R_i \setminus C') \cup R^i).$$

Likewise, expand $\alpha_{i-1}(S \cup \{i\}, R \setminus \{i\})$ using the induction hypothesis into

$$\sum_{S \subseteq C' \subseteq (S \cup R)_{i-1}} \alpha(C' \cup S^i \cup \{i\}, (R_i \setminus C') \cup (R \setminus \{i\}).$$

where we write $C'$ for $C \cup \{i\}$. Observe that this sum equals

$$\sum_{S \subseteq C' \subseteq (S \cup R)_{i-1}} \alpha(C' \cup S^i, (R_i \setminus C') \cup R^i),$$

because $i \notin S$ and $i \in R$. Adding up the obtained two sums over $C'$ yields

$$\alpha_i(S, R) = \sum_{S \subseteq C' \subseteq (S \cup R)_i} \alpha(C' \cup S^i, (R_i \setminus C') \cup R^i),$$

which matches the induction hypothesis (5). \qed

B PROOF OF LEMMA 4

In order to prove Lemma 4, we first prove the following lemma:

**Lemma 9.** Backtracking starting from $g(C, U)$ makes at most $|U|$ recursive nonterminating visits to $g$ (including the visit to $g(C, U)$).

**Proof.** We show the claim by induction over $|U|$. The case $|U| = 0$ is trivial as it terminates. Suppose that $|U| \geq 1$ and
the claim holds for smaller $U$. The visit to $g(C, U)$ is followed by recursive visits to (i) $g(C, U \setminus R)$ and (ii) $h(C, R)$. By the induction assumption (i) amounts to at most $|U| - |R|$ recursive nonterminating visits to $g$. Visit (ii) is followed by a visit to $f$ succeeded by a visit to $g(C', R \setminus C')$ for some $C'$ with $R \cap C' \neq \emptyset$. Thus, by the induction assumption, (ii) amounts to at most $|R| - 1$ recursive nonterminating visits to $g$. The total, including the visit to $g(C, U)$, is thus at most $(|U| - |R|) + (|R| - 1) + 1 = |U|$. \hfill \qed

Now we can prove Lemma 4:

**Proof of Lemma 4.** Observe that the first two visits are to $f(\emptyset, V)$ and to $g(C, U)$ where $|U| \leq |V| - 1$. By Lemma 9, there are thus at most $n - 1$ nonterminating visits to $g$. Also note that a visit to $h$ is always from a nonterminating visit to $g$ and a visit to $f$ always from a visit to $h$ (except the first visit). The result follows. \hfill \qed

## C PROOF OF LEMMA 5

In order to prove Lemma 5, we first prove the following lemma:

**Lemma 10.** Consider backtracking from $g(C, U)$ onwards. Let $\{(C_1, U_1), (C_2, U_2), \ldots, (C_d, U_d)\}$ be the set pairs of the recursive nonterminating visits to $g$, including $(C, U)$. Then there exists an ordering of the $d$ set pairs such that

$$|C_i| + |U_i| \leq |C| + |U| - i + 1 \text{ for all } i = 1, \ldots, d.$$

**Proof.** We show the claim by induction over $|U|$. The case $|U| = 0$ is trivial as there are no recursive visits. Suppose that $|U| \geq 1$ and the claim holds for smaller $U$. The visit to $g(C, U)$ is followed by recursive visits to (i) $g(C, U \setminus R)$ and (ii) $h(C, R)$.

First, let $(C_1, U_1) = (C, U)$. Clearly then the claim holds for $i = 1$.

Let then $\{(C_2, U_2), \ldots, (C_{d'}, U_{d'})\}$ be the $d'$ set pairs of $g$ visited in branch (i). By the induction assumption and the fact that $U \cap R \neq \emptyset$, there exists an ordering over these set pairs such that for all $i = 2, \ldots, d'$,

$$|C_i| + |U_i| \leq |C| + |U \setminus R| - (i - 1) + 1 \leq |C| + |U| - i + 1.$$

Thus the claim holds for $i = 2, \ldots, d' + 1$.

Finally, branch (ii) makes, via $h$ and $f$, a recursive visit to $g(C', R \setminus C')$ for some $C'$ with $R \cap C' \neq \emptyset$. Let $\{(C_{d'+2}, U_{d'+2}), \ldots, (C_{d'+d''+1}, U_{d'+d''+1})\}$ be the $d''$ set pairs of $g$ visited in branch (ii). By the induction assumption, there exists an ordering over these set pairs such that for all $i = d' + 2, \ldots, d' + d'' + 1$,

$$|C_i| + |U_i| \leq |C'| + |R \setminus C'| - (i - d' - 1) + 1 \leq (|C| + |R| - 1) - i + (|U| - |R|) + 2 = |C| + |U| - i + 1.$$

The second inequality uses the fact that $C' \cup R = S \cup R \subset C \cup R$, where $\hat{S}$ is selected during the visit to $h$, and the fact that by Lemma 9, $d'' \leq |U \setminus R| = |U| - |R|$. As $d' + d'' + 1 = d$, the claim thus holds for $i = d' + 1, \ldots, d$, which completes the proof. \hfill \qed

Now we can prove Lemma 5:

**Proof of Lemma 5.** For $g$ the claim directly follows by applying Lemma 10. As a visit to $h(C, R)$ is always from a nonterminating visit to $g(C, U)$ with some $U$ such that $C \cup R \subseteq C \cup U$, the claim follows for $h$. Finally, except the first visit, any other visit to $f(S, R)$ is always from a visit to $h(C, R)$ with some $C$ such that $|S \cup R| \leq |C \cup R| - 1$. Thus we can index the remaining set pairs from 2 to $d_f$ such that the claim holds for them. Then $(S_1, R_1)$ must be set to $(\emptyset, V)$ so that $|S_1| + |R_1| \leq n$ and the claim follows also for $i = 1$. \hfill \qed