Supplementary Material:
Max-Product Belief Propagation for Linear Programming

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Proof of Corollary 6

Since LP (11) always has an integral solution, it suffices to show that the max-product BP on GM (12) converges to the solution of LP. The proof of Corollary 6 can be done by using Theorem 1. From GM (12), each variable is connected to two factors (Condition *C2*). Now, lets check Condition *C3*. Suppose there are v and $x_{\delta(v)}$ with $\psi_v(x_{\delta(v)}) = 1$. Consider the case when there is $e \in \delta^i(v)$ with $x_e = 1 \neq x^*_e$. If $e' \in \delta^i(v)$ with $x_{e'} = 0 \neq x^*_{e'}$ exists, choose such e'. Otherwise, choose $e' \in \delta^o(v)$ with $x_{e'} = 1 \neq x^*_{e'}$. On the other hand, consider when there is $e \in \delta^i(v)$ with $x_e = 0 \neq x^*_e$. If $e' \in \delta^o(v)$ with $x_{e'} = 1 \neq x^*_{e'}$ exists, choose such e'. If not, choose $e' \in \delta^i(v)$ with $x_{e'} = 0 \neq x^*_{e'}$. Then,

$$\psi_v(x'_{\delta(v)}) = 1, \qquad \text{where } x'_{e''} = \begin{cases} x_{e''} & \text{if } e'' \neq e, e' \\ x^*_{e''} & \text{otherwise} \end{cases}.$$

$$\psi_v(x''_{\delta(v)}) = 1, \qquad \text{where } x''_{e''} = \begin{cases} x_{e''} & \text{if } e'' = e, e' \\ x^*_{e''} & \text{otherwise} \end{cases}.$$

We can apply similar argument for the case when $e \in \delta^o(v)$, $v = s$ or t. From Theorem 1, we can conclude that if the solution of LP (11) is unique, the max-product BP on GM (12) converges to the solution of LP (11).

Proof of Corollary 7

The proof of Corollary 7 can be done by using Theorem 1. From GM (15), each variable is connected to two factors (Condition *C2*). Now, lets check Condition *C3*. Suppose there are v and $x_{\delta(v)}$ with $\psi_v(x_{\delta(v)}) = 1$. Consider the case when there is $e_i \in \delta(v)$ with $x_{e_i} = 1 \neq x^*_{e_i}$. Then, there is $e'_j \in \delta(v)$ with $x_{e'_j} = 0 \neq x^*_{e'_j}$. Choose such e'_j. On the other hand, consider when there is $e_i \in \delta(v)$ with $x_{e_i} = 0 \neq x^*_{e_i}$. Then, there is $e'_j \in \delta(v)$ with $x_{e'_j} = 1 \neq x^*_{e'_j}$. Choose such e'_j. Then,

$$\psi_v(x'_{\delta(v)}) = 1, \qquad \text{where } x'_{e''_k} = \begin{cases} x_{e''_k} & \text{if } e''_k \neq e_i, e'_j \\ x^*_{e''_k} & \text{otherwise} \end{cases}.$$

$$\psi_v(x''_{\delta(v)}) = 1, \qquad \text{where } x''_{e''_k} = \begin{cases} x_{e''_k} & \text{if } e''_k = e_i, e'_j \\ x^*_{e''_k} & \text{otherwise} \end{cases}.$$

From Theorem 1, we can conclude that if the solution of LP (14) is unique, the max-product BP on GM (15) converges to the solution of LP (14).

Proof of Corollary 8

From GM (18), each variable is connected to two factors (Condition *C2*). Now, lets check Condition *C3*. For $v \in V$, we can apply same argument as the maximum weight matching case. Suppose there are v_C and $y_{\delta(v_C)}$ with $\psi_C(y_{\delta(v_C)}) = 1$. Consider the case when there is $(u_1, v_C) \in \delta(v_C)$ with $y_{(u_1,v_C)} = 1 \neq y^*_{(u_1,v_C)}$. As a feasible solution $y_{\delta(v_C)}$ forms a disjoint even paths (Shin et al., 2013), check edges along the path contains u_1. If there is $u_2 \in V(C)$ in the path with $y_{(u_2,v_C)} = 1 \neq y^*_{(u_2,v_C)}$ exists, choose such (u_1, v_C). Otherwise, choose $(u_2, v_C) \in V(C)$ with $y_{(u_2,v_C)} = 0 \neq y^*_{(u_2,v_C)}$ at the end of the path. On the other hand, consider the case when there is $(u_1, v_C) \in \delta(v_C)$ with $y_{(u_1,v_C)} = 0 \neq y^*_{(u_1,v_C)}$. As a feasible solution $y_{\delta(v_C)}$ form a disjoint even paths, check edges along the path contains u_1. If there is $u_2 \in V(C)$ in the path with $y_{(u_2,v_C)} = 0 \neq y^*_{(u_2,v_C)}$ exists, choose such (u_1, v_C). Otherwise, choose $(u_2, v_C) \in V(C)$ with $y_{(u_2,v_C)} = 1 \neq y^*_{(u_2,v_C)}$ at the end of the path. Then, from disjoint even paths point of view, we can check that

$$\psi_C(y'_{\delta(v_C)}) = 1, \quad \text{where } y'_{(u,v_C)} = \begin{cases} y_{(u,v_C)} & \text{if } u \neq u_1, u_2 \\ y^*_{(u,v_C)} & \text{otherwise} \end{cases}.$$

$$\psi_C(y''_{\delta(v_C)}) = 1, \quad \text{where } y''_{(u,v_C)} = \begin{cases} y_{(u,v_C)} & \text{if } u = u_1, u_2 \\ y^*_{(u,v_C)} & \text{otherwise} \end{cases}.$$

From Theorem 1, we can conclude that if the solution of LP (17) is unique and integral, the max-product BP on GM (18) converges to the solution of LP (17).

Proof of Corollary 9

The proof of Corollary can be done by using Theorem 1. From GM (22), each variable is connected to two factors (Condition *C2*). Now, lets check Condition *C3*. Suppose there are v and $x_{\delta(v)}$ with $\psi_v(x_{\delta(v)}) = 1$. Consider the case when there is $e_i \in \delta(v)$ with $x_{e_i} = 1 \neq x^*_{e_i}$. If there is $e'_j \in \delta(v)$ with $x_{e'_j} = 0 \neq x^*_{e'_j}$, choose such e'_j. Otherwise, choose $e'_j = e_i$ On the other hand, consider when there is $e_i \in \delta(v)$ with $x_{e_i} = 0 \neq x^*_{e_i}$. If there is $e'_j \in \delta(v)$ with $x_{e'_j} = 1 \neq x^*_{e'_j}$, choose such e'_j. Otherwise, choose $e'_j = e_i$ Then,

$$\psi_v(x'_{\delta(v)}) = 1, \qquad \text{where } x'_{e''_k} = \begin{cases} x_{e''_k} & \text{if } e''_k \neq e_i, e'_j \\ x^*_{e''_k} & \text{otherwise} \end{cases}.$$

$$\psi_v(x''_{\delta(v)}) = 1, \qquad \text{where } x''_{e''_k} = \begin{cases} x_{e''_k} & \text{if } e''_k = e_i, e'_j \\ x^*_{e''_k} & \text{otherwise} \end{cases}.$$

From Theorem 1, we can conclude that if the solution of LP (21) is unique, the max-product BP on GM (22) converges to the solution of LP (21).

Proof of Corollary 10

The proof of Corollary 10 can be done by using Theorem 1. From GM (24), each variable is connected to two factors (Condition *C2*). Now, lets check Condition *C3*. Suppose there are v and $x_{\delta(v)}$ with $\psi_v(x_{\delta(v)}) = 1$. Consider the case when there is $e \in \delta(v)$ with $x_e = 1 \neq x^*_e$. By formulation of GM, there exists $e' \in \delta(v)$ with $x_{e'} = 0 \neq x^*_{e'}$. Choose such e'. On the other hand, consider when there is $e \in \delta(v)$ with $x_e = 0 \neq x^*_e$. There exists $e' \in \delta(v)$ with $x_{e'} = 1 \neq x^*_{e'}$. Choose such e'. Then,

$$\psi_v(x'_{\delta(v)}) = 1, \qquad \text{where } x'_{e''} = \begin{cases} x_{e''} & \text{if } e'' \neq e, e' \\ x^*_{e''} & \text{otherwise} \end{cases}.$$

$$\psi_v(x''_{\delta(v)}) = 1, \qquad \text{where } x''_{e''} = \begin{cases} x_{e''} & \text{if } e'' = e, e'' \\ x^*_{e''} & \text{otherwise} \end{cases}.$$

From Theorem 1, we can conclude that if the solution of LP (23) is unique and integral, the max-product BP on GM (24) converges to the solution of LP (23).

Proof of Corollary 11

The proof of Corollary 11 can be done by using Theorem 1. From GM (26), each variable is connected to two factors (Condition *C2*). Now, lets check Condition *C3*. Suppose there are v and $x_{\delta(v)}$ with $\psi_v(x_{\delta(v)}) = 1$. Consider the case when there is $e \in \delta(v)$ with $x_e = 1 \neq x^*_e$. If $e' \in \delta(v)$ with $x_{e'} = 0 \neq x^*_{e'}$ exists. Choose such e'. Otherwise, there exists $e' \in \delta(v)$ with $x_{e'} = 1 \neq x^*_{e'}$. Choose such e'. On the other hand, consider when there is $e \in \delta(v)$ with $x_e = 0 \neq x^*_e$. If $e' \in \delta(v)$ with $x_{e'} = 1 \neq x^*_{e'}$ exists. Choose such e'. Otherwise, there exists $e' \in \delta(v)$ with $x_{e'} = 0 \neq x^*_{e'}$. Choose such e'. Then,

$$\psi_v(x'_{\delta(v)}) = 1, \qquad \text{where } \left\{ \begin{array}{l} x'_{e''} = \begin{cases} x_{e''} & \text{if } e'' \neq e, e' \\ x^*_{e''} & \text{otherwise} \end{cases} \\ y'_v = y^*_v \end{array} \right. .$$

$$\psi_v(x''_{\delta(v)}) = 1, \qquad \text{where } \left\{ \begin{array}{l} x''_{e''} = \begin{cases} x_{e''} & \text{if } e'' = e, e'' \\ x^*_{e''} & \text{otherwise} \end{cases} \\ y'_v = y_v \end{array} \right. .$$

Case of y variable can be done in similar manner. From Theorem 1, we can conclude that if the solution of LP (25) is unique and integral, the max-product BP on GM (26) converges to the solution of LP (25).