Non-parametric causal models II.

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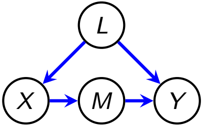
UAI Tutorial
12th July 2015

Model

- So far we have shown how to estimate interventional distributions separately, but no guarantee these estimates are coherent.
- We also may have multiple identifying expressions: which one should we use?



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back door?
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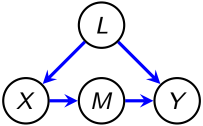

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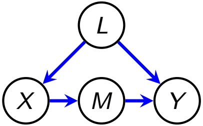

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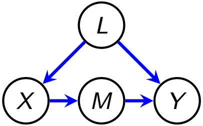

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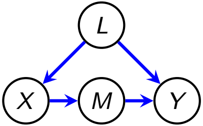

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All this suggests we should define a model which we can parameterize.

Outline

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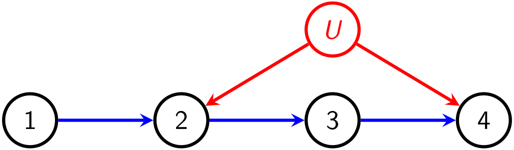
Ancestral Sets



$$p(x_1, x_2, x_3, x_4)$$
$$= \sum_u p(u)\, p(x_1)\, p(x_2 \mid x_1, u)\, p(x_3 \mid x_2)\, p(x_4 \mid x_3, u)$$

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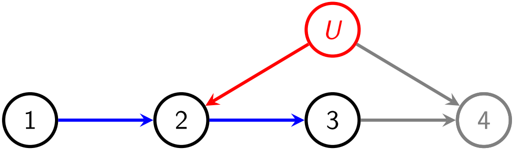

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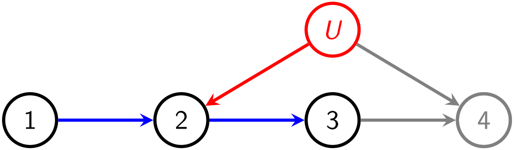

$$
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&p(x_1, x_2, x_3) \\
&= \sum_{x_4} \sum_{u} p(u)\, p(x_1)\, p(x_2 \mid x_1, u)\, p(x_3 \mid x_2)\, p(\mathbf{x_4} \mid x_3, u) \\
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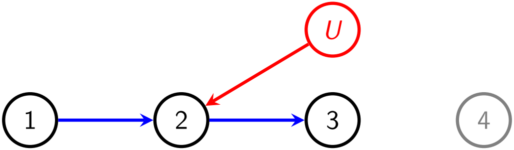

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&= \sum_{\boldsymbol{u}} p(\boldsymbol{u})\, p(x_1)\, p(x_2 \mid x_1, \boldsymbol{u})\, p(x_3 \mid x_2)
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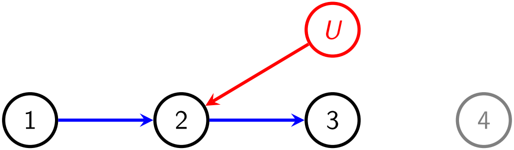

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&= p(x_1)\, p(x_3 \mid x_2)\, p(x_2 \mid x_1).
\end{aligned}
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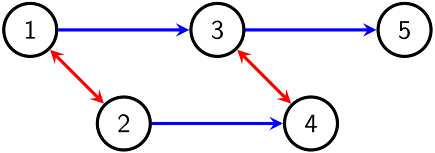
Density has form corresponding to ancestral sub-graph.

Factorization into Districts

District is a maximal set connected by latent variables / bidirected edges:



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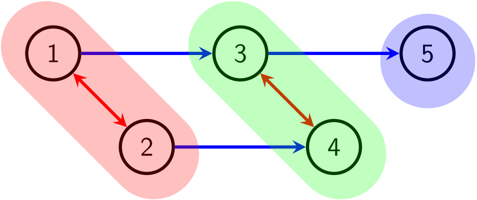

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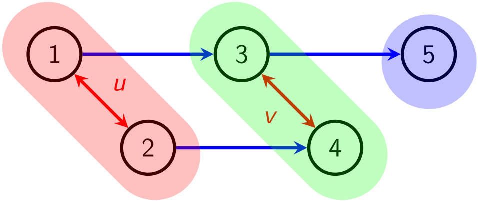

$$\sum_{u,v} p(u)\, p(x_1 \mid u)\, p(x_2 \mid u) \quad p(v)\, p(x_3 \mid x_1, v)\, p(x_4 \mid x_2, v) \quad p(x_5 \mid x_3)$$

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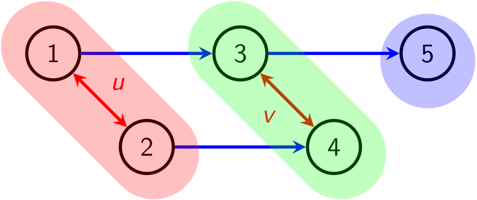

$$\sum_{u,v} \boxed{p(u)\, p(x_1 \mid u)\, p(x_2 \mid u)} \; \boxed{p(v)\, p(x_3 \mid x_1, v)\, p(x_4 \mid x_2, v)} \; \boxed{p(x_5 \mid x_3)}$$

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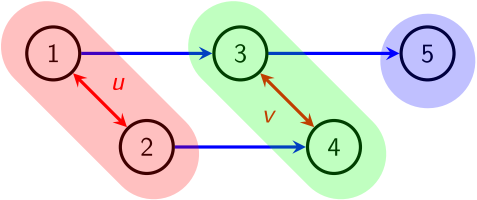

$$\sum_{u,v} p(u)\, p(x_1 \mid u)\, p(x_2 \mid u)\; p(v)\, p(x_3 \mid x_1, v)\, p(x_4 \mid x_2, v)\; p(x_5 \mid x_3)$$

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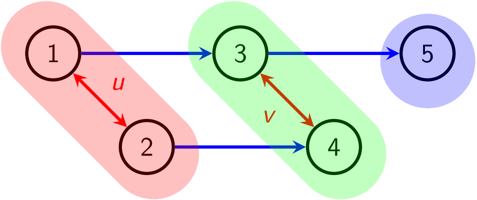

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$$= \boxed{q_{12}(x_1, x_2)} \cdot \boxed{q_{34}(x_3, x_4 \mid x_1, x_2)} \cdot \boxed{q_5(x_5 \mid x_3)}.$$

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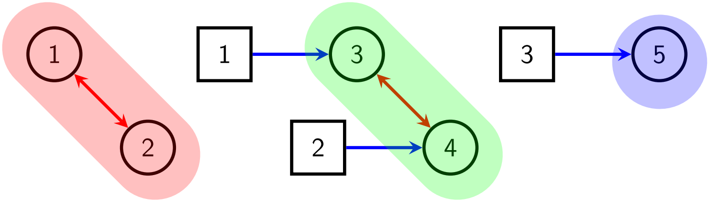

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$$= \prod_i q_{D_i}(x_{D_i} \mid x_{\mathrm{pa}(D_i)\setminus D_i})$$

Each q_D piece should come from the model based on district subgraph and its parents ($\mathcal{G}[D]$).

Axiomatic Approach

We use these two rules to define our model.

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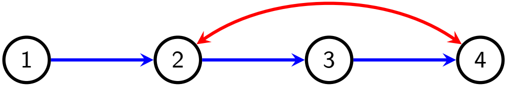
Note that one can iterate between 1 and 2.

This defines the **nested Markov model** $\mathcal{N}(\mathcal{G})$.

Verma Example



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and therefore $X_1 \perp\!\!\!\perp X_3 \mid X_2$.

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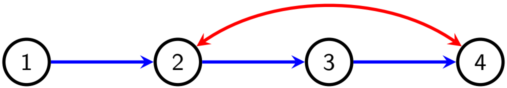

Axiom 2:

$$p(x_1, x_2, x_3, x_4) = q_1(x_1) \cdot q_3(x_3 \mid x_2) \cdot q_{24}(x_2, x_4 \mid x_1, x_3).$$

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This places a non-trivial constraint on p.

Relationship to Fixing

Could also recursively define a model $\mathcal{N}'$ by fixing:

$$p \in \mathcal{N}'(\mathcal{G}) \implies \phi_v(p) \in \mathcal{N}'(\phi_v(\mathcal{G}))$$

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The recursive factorization model is useful for parameterization proofs.

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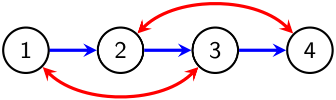
Recall that to 'fix' a vertex, it must not have children in its district.
Equivalent to splitting, marginalizing, and then pasting back together.



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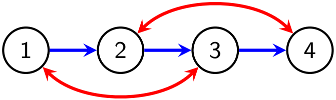
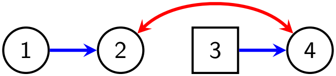


fix $\rightarrow$



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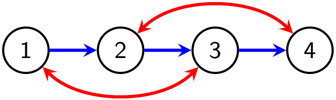

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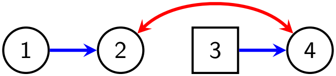
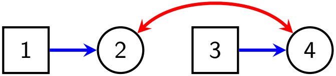
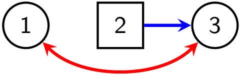


split into districts $\downarrow$





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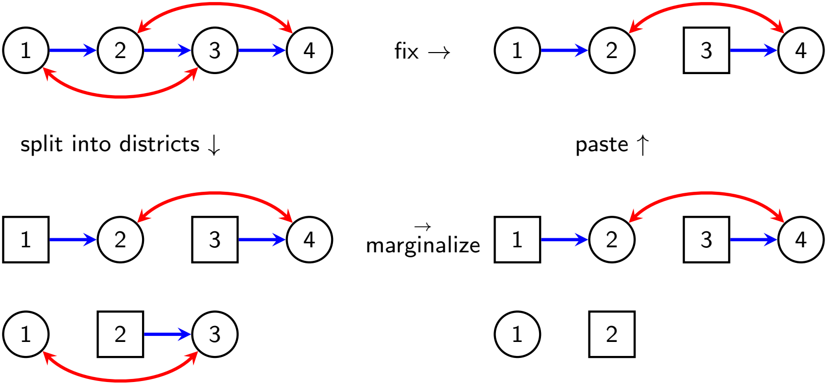
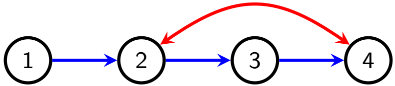


Notations



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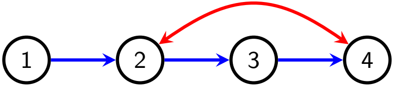

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Theorem (Richardson, Shpitser, Robins, 201x)

For a positive distribution $p \in \mathcal{N}(\mathcal{G})$ and vertices v_1, v_2 that are fixable in $\mathcal{G}$,

$$(\phi_{v_1} \circ \phi_{v_2})(p) = (\phi_{v_2} \circ \phi_{v_1})(p).$$

Hence, the order of fixing doesn't matter.

This is another way of saying that all identifying expressions for a causal quantity will be the same.

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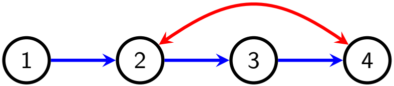

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For any reachable R this justifies the (unambiguous) notation $\phi_{V \setminus R}$.

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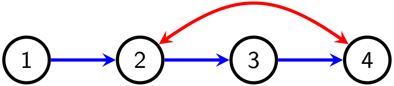

Note that we can potentially reach the same district by different methods: e.g. marginalize 4, fix 1, 2 or reverse.

Theorem (Richardson, Shpitser, Robins, 201x)

For a positive distribution $p \in \mathcal{N}(\mathcal{G})$ and vertices v_1, v_2 that are fixable in $\mathcal{G}$,

$$(\phi_{v_1} \circ \phi_{v_2})(p) = (\phi_{v_2} \circ \phi_{v_1})(p).$$

Hence, the order of fixing doesn't matter.

This is another way of saying that all identifying expressions for a causal quantity will be the same.

For any reachable R this justifies the (unambiguous) notation $\phi_{V \setminus R}$.

For $p \in \mathcal{N}(\mathcal{G})$, let

$$\mathcal{G}[R] \equiv \phi_{V \setminus R}(\mathcal{G}) \qquad\qquad q_R \equiv \phi_{V \setminus R}(p).$$

Reachable CADMGs

Note that $\mathcal{G}[R]$ is always just the CADMG with:

- random vertices R,
- fixed vertices $\text{pa}_{\mathcal{G}}(R) \setminus R$,
- induced edges from $\mathcal{G}$ among R and of the form $\text{pa}_{\mathcal{G}}(R) \to R$.

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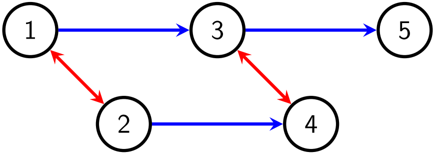

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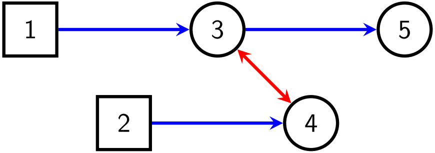

Graph shown is $\mathcal{G}[\{3, 4, 5\}]$.

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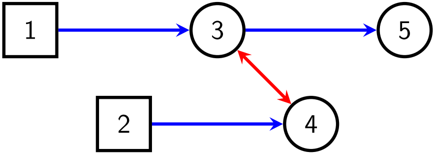
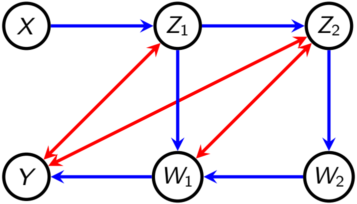


Graph shown is $\mathcal{G}[\{3, 4, 5\}]$.

Also recall that

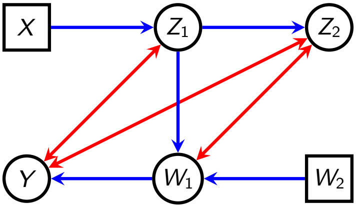
$$q_R(x_R \mid x_{\text{pa}(R) \setminus R}) = p(x_R \mid do(x_{V \setminus R})).$$

Example



$p(x, y, w_1, w_2, z_1, z_2)$

Example



$$q_{yw_1z_1z_2}(y, w_1, z_1, z_2 \mid x, w_2) = \frac{p(x, y, w_1, w_2, z_1, z_2)}{p(x)p(w_2 \mid z_2)}$$

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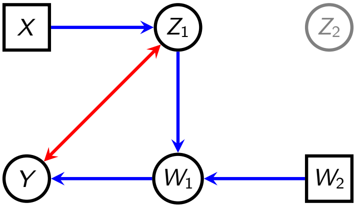

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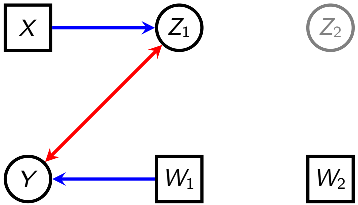

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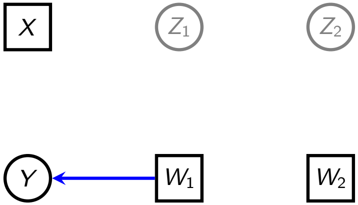

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and $q_{yz_1}(y \mid x, w_1)$ doesn't depend upon x.

Nested Markov Model

Various equivalent formulations:

Factorization into Districts.

For each reachable R in $\mathcal{G}$,

$$q_R(x_R \mid x_{\mathrm{pa}(R)\setminus R}) = \prod_{D \in \mathcal{D}(\mathcal{G}[R])} f_D(x_{D \cup \mathrm{pa}(D)})$$

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Weak Global Markov Property.
For each reachable R in $\mathcal{G}$,

$$A \text{ m-separated from } B \text{ by } C \text{ in } \mathcal{G}[R] \implies X_A \perp\!\!\!\perp X_B \,|\, X_C[q_R].$$

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Ordered Local Markov Property.
For every intrinsic S and v maximal in S under some topological ordering,

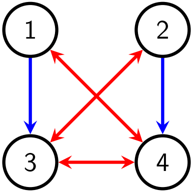
$$X_v \perp\!\!\!\perp X_{V \setminus \mathrm{mb}_{\mathcal{G}[S]}(v)} \mid X_{\mathrm{mb}_{\mathcal{G}[S]}(v)} \, [q_S].$$

Theorem. These are all equivalent.

Outline

Heads and Tails

As established, we can factorize a graph into districts; however, finer factorizations are possible.



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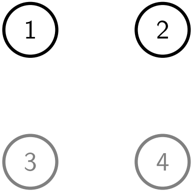

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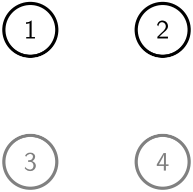

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So could factorize this as

$$\begin{aligned} p(x_1, x_2, x_3, x_4) &= p(x_1, x_2) p(x_3, x_4 \mid x_1, x_2) \\ &= p(x_1) p(x_2) p(x_3, x_4 \mid x_1, x_2). \end{aligned}$$

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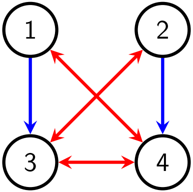

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Note that the vertices $\{3, 4\}$ can't be d-separated from one another.

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$$p(x_S \,|\, x_{\mathrm{pa}(S)\setminus S}) = p(x_H \,|\, x_T) \cdot p(x_{S\setminus H} \,|\, x_{\mathrm{pa}(S)\setminus S}).$$

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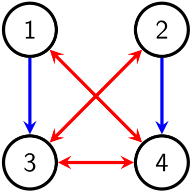

But vertices in $S \setminus H$ may factorize:

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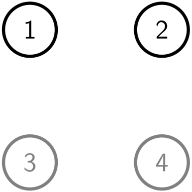

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&= p(x_3, x_4 \mid x_1, x_2) p(x_1) p(x_2).
\end{aligned}$$

Factorizations

Recursively define a partition of reachable sets as follows. If R has multiple districts,

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Theorem (Head Factorization Property)

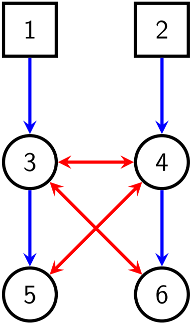
p obeys the nested Markov property for $\mathcal{G}$ if and only if for every reachable set R,

$$q_R(x_R \,|\, x_{\mathrm{pa}(R)\setminus R}) = \prod_{H \in [R]_{\mathcal{G}}} q_H(x_H \,|\, x_T).$$

Here $q_H \equiv q_{S(H)}$ is density associated with intrinsic set for H. (Recursive heads are in one-to-one correspondence with intrinsic sets.)

Heads and Tails

Recall, intrinsic sets are reachable districts:



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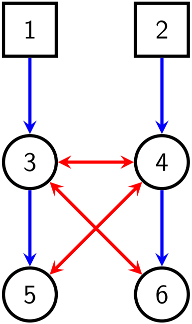

intrinsic set	I	$\{3, 4, 5, 6\}$
recursive head	H	$\{5, 6\}$
tail	T	$\{1, 2, 3, 4\}$

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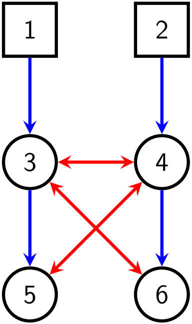

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intrinsic set	I	$\{3, 4\}$
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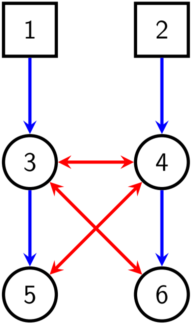

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So

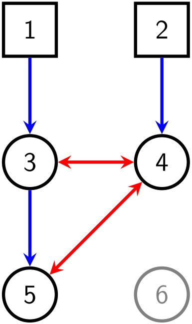
$$[\{3, 4, 5, 6\}]_{\mathcal{G}} = \{\{3, 4\}, \{5, 6\}\}.$$

Factorization:

$$q_{3456}(x_{3456} \mid x_{12}) = q_{56}(x_{56} \mid x_{1234}) \cdot q_{34}(x_{34} \mid x_{12})$$

Heads and Tails

What if we fix 6 first?



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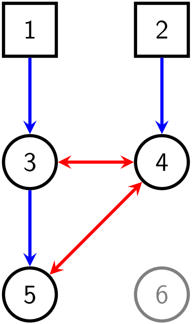

intrinsic set	I	$\{3, 4, 5\}$
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tail	T	$\{1, 2, 3\}$

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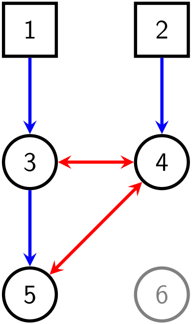

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tail	T	$\{1, 2, 3\}$
intrinsic set	I	$\{3\}$
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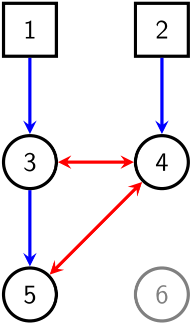

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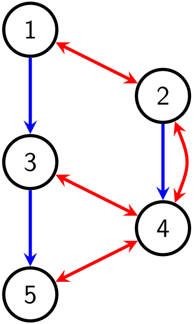
So

$$[\{3,4,5\}]_{\mathcal{G}} = \{\{3\},\{4,5\}\}.$$

Factorization:

$$q_{345}(x_{345} \mid x_{12}) = q_{45}(x_{45} \mid x_{123}) \cdot q_3(x_3 \mid x_1)$$

Heads and Tails



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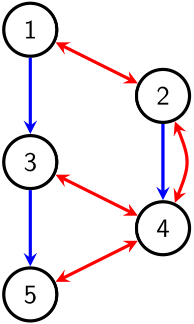

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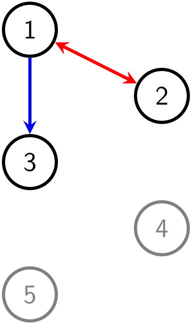

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intrinsic set	I	$\{1, 2\}$
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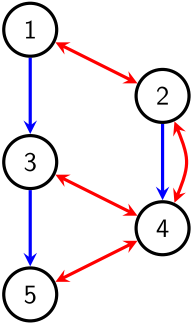
Heads and Tails



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tail	T	$\{1\}$

Factorization:

$$q_{12345}(x_{12345}) = q_{45}(x_{45} \mid x_{123}) \cdot q_3(x_3 \mid x_1) \cdot q_{12}(x_{12}).$$

Outline

Parameterizations

Let $\mathcal{M}$ be a model (i.e. collection of probability distributions).

A **parameterization** is a continuous bijective map

$$\theta : \mathcal{M} \to \Theta$$

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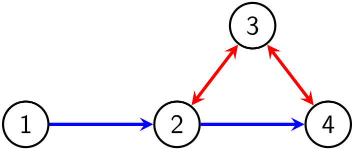
We will assume all variables are binary; this extends easily to the general categorical / discrete case.

Factorization into Districts



We'd like a parametrization which exhibits the axioms directly.
Then all reachable subgraphs will be taken care of too.

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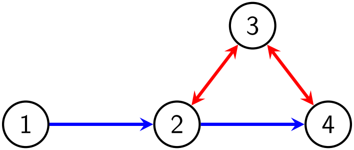

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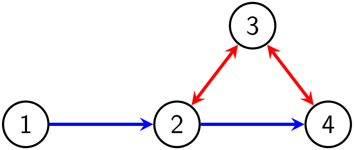

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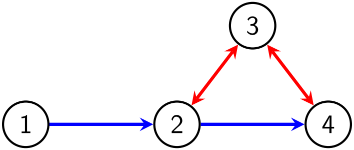
The Game: proceed inductively to explicitly construct θ, and assume all reachable sub-graphs can be parameterized.

If $\mathcal{G}$ has multiple districts $\mathcal{D}$, then by Axiom 1

$$p(x_V \mid x_W) = \prod_{D \in \mathcal{D}(\mathcal{G})} q_D(x_D \mid x_{\mathrm{pa}(D) \setminus D});$$

so parameterize each q_D according to $\mathcal{G}[D]$ separately (parameter cut).

Factorization into Districts



We'd like a parametrization which exhibits the axioms directly.
Then all reachable subgraphs will be taken care of too.

The Game: proceed inductively to explicitly construct θ, and assume all reachable sub-graphs can be parameterized.

If $\mathcal{G}$ has multiple districts $\mathcal{D}$, then by Axiom 1

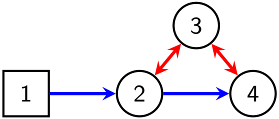
$$p(x_V \mid x_W) = \prod_{D \in \mathcal{D}(\mathcal{G})} q_D(x_D \mid x_{\mathrm{pa}(D) \setminus D});$$

so parameterize each q_D according to $\mathcal{G}[D]$ separately (parameter cut).
E.g.

$$p(x_1, x_2, x_3, x_4) = p(x_1) \cdot p(x_2, x_3, x_4 \mid x_1).$$

Note for a DAG this is usual CPTs.

Marginalization



To satisfy Axiom 2, we'd like ancestral margins of $p(x_{234} \mid x_1)$ to factorize according to CADMG.

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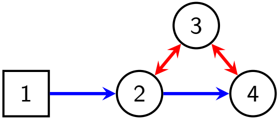

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$$p(x_2, x_3, 1_4 \mid x_1) + p(x_2, x_3, 0_4 \mid x_1) = p(x_2, x_3 \mid x_1)$$

and $p(x_{23} \mid x_1)$ should by parameterized according to $\mathcal{G}_{-4}$.

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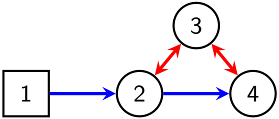

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Can use this to define probabilities where some entries of head are 1.

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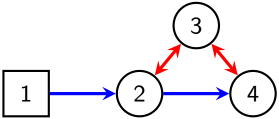

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Repeat until all vertices in recursive head are 0; e.g.

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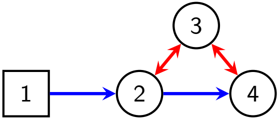

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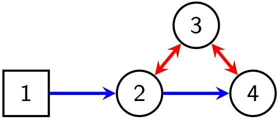
So every term represents an ancestral sub-graph, except for final term where every variable in the recursive head is 0.

Example



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We're now 'stuck' precisely when we get a full head of 0s.

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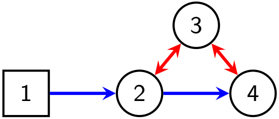

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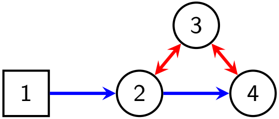

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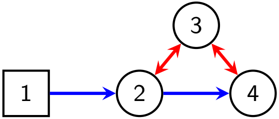

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Have a collection of parameters $\theta_{34}(x_1, x_2)$ associated with the head $H = \{3, 4\}$ conditional upon the tail $\{1, 2\}$.

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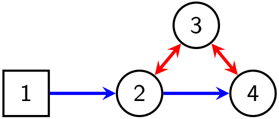

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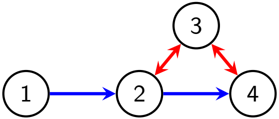
Generally parameters are

$$\theta_H(x_T) \equiv q_H(0_H \mid x_T), \qquad \text{for all heads } H, x_T.$$

Probabilities



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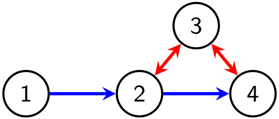

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First,

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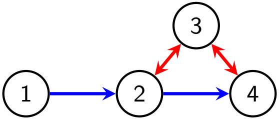

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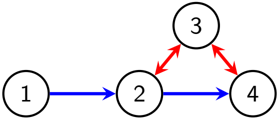

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For the district $\{2, 3, 4\}$ get

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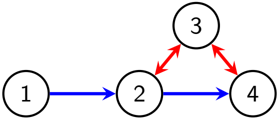

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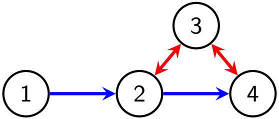

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\end{aligned}$$

Putting this all together:

$$\begin{aligned}
&p(1_1, 0_2, 1_3, 1_4) \\
&= \{1 - \theta_1\}\{\theta_2(1) - \theta_{23}(1) - \theta_2(1)\theta_4(0) + \theta_2(1)\theta_{34}(1, 0)\}.
\end{aligned}$$

Parameterization

Say binary distribution p *parameterized according to* $\mathcal{G}$ if[1]

$$p(x_V \mid x_W) = \sum_{O \subseteq C \subseteq V} (-1)^{|C \setminus O|} \prod_{H \in [C]_{\mathcal{G}}} \theta_H(x_T),$$

for some parameters $q_H(x_T)$ where $O = \{v : x_v = 0\}$.

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Note: there is no need to assume that $\theta_H(x_T) \in [0, 1]$, this comes for free if $p(x_V \mid x_W) \geq 0$.

If suitable causal interpretation of model exists,

$$\begin{aligned} \theta_H(x_T) = q_S(0_H \mid x_T) &= p(0_H \mid x_{S \setminus H}, do(x_{T \setminus S})) \\ &\neq p(0_H \mid x_T). \end{aligned}$$

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Theorem (Evans and Richardson, forthcoming)

p is parameterized according to $\mathcal{G}$ if and only if it recursively factorizes according to $\mathcal{G}$ (so $p \in \mathcal{N}(\mathcal{G})$).

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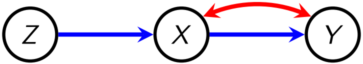
Example 1



Intrinsic Sets	Z	X, Y	X

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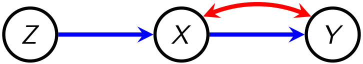

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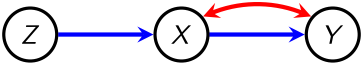

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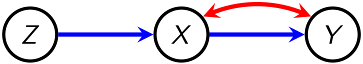

Intrinsic Sets	Z	X, Y	X
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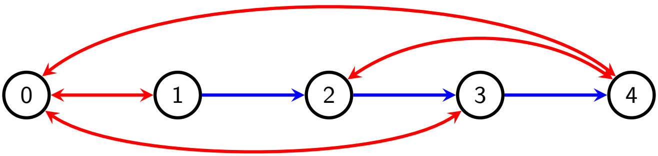
So parameterization is just

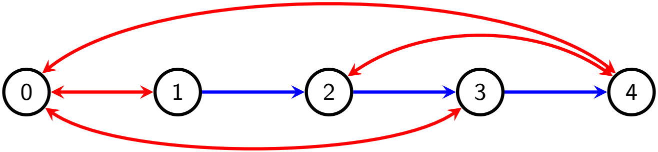
$$p(z = 0), \qquad p(x = 0 \mid z) \qquad p(y = 0 \mid x, z).$$

Model is saturated.

Example 2



Example 2



$$p(0_0, 1_1, 1_2, 0_3, 0_4) = p(0_0, 1_1, 1_2, 0_3) \cdot q_4(0_4 \mid 0_0, 1_1, 1_2, 0_3)$$

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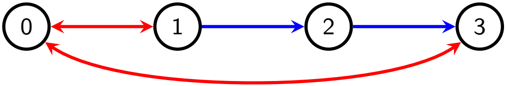

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$$p(0_0, 1_1, 1_2, 0_3) = q_2(1_2 \mid 1_1) \cdot q_{013}(0_0, 1_1, 0_3 \mid 1_2)$$

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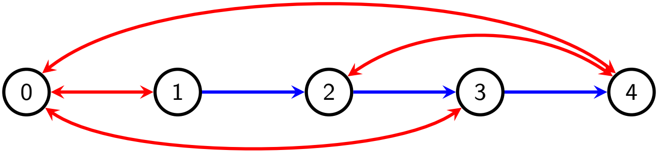

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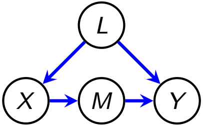
so

$$
p(0_0, 1_1, 1_2, 0_3, 0_4) = \{1 - \theta_2(1)\} \{\theta_{03}(1) - \theta_{013}(1)\} \cdot \theta_4(0, 1, 1, 0).
$$

Model

- So far we have shown how to estimate interventional distributions separately, but no guarantee these estimates are coherent.
- We also may have multiple identifying expressions: which one should we use?



$p(Y \mid do(X))$
front door?
back door?
does it matter?

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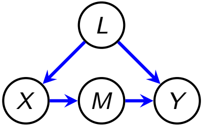

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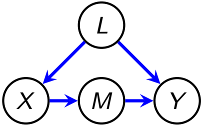

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- We can test constraints separately, but ultimately don't have a way to check if the model is a good one.
- Being able to evaluate a likelihood would allow lots of standard inference techniques (e.g. LR, Bayesian).

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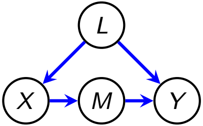

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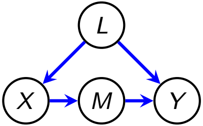

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All this suggests we should define a model which we can parameterize.

Outline

Exponential Families

Theorem

Let $\mathcal{N}(\mathcal{G})$ be the collection of binary distributions that recursively factorize according to $\mathcal{G}$. Then $\mathcal{N}(\mathcal{G})$ is a curved exponential family of dimension

$$d(\mathcal{G}) = \sum_{H \in \mathcal{H}(\mathcal{G})} 2^{|\operatorname{tail}(H)|}.$$

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Can also parameterize as GLM response model (Shpitser et al., 2013).

Algorithms for Model Search

Can, for example, use greedy edge replacement for a score-based approach (Evans and Richardson, 2010).

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Can, for example, use greedy edge replacement for a score-based approach (Evans and Richardson, 2010).

Shpitser et al. (2011) developed efficient algorithms for evaluating probabilities in the alternating sum.

Currently no equivalent of PC algorithm for full nested model.

Can use FCI algorithm (Spirtes at al., 2000) for **ordinary mixed graphical models** (conditional independences only), which is generally a supermodel of nested (see Evans and Richardson, 2014).

Parameterization References

Evans – Graphs for margins of Bayesian networks, *arXiv:1408.1809*, 2014.

Evans and Richardson – Maximum likelihood fitting of acyclic directed mixed graphs to binary data. *UAI*, 2010.

Evans and Richardson – Markovian acyclic directed mixed graphs for discrete data. *Annals of Statistics*, 2014.

Shpitser, Richardson, Robins. An efficient algorithm for computing interventional distributions in latent variable causal models. *UAI*, 2011.

Shpitser, Richardson, Robins and Evans – Parameter and structure learning in nested Markov models. *UAI*, 2012.

Shpitser, Evans, Richardson, and Robins – Sparse nested Markov models with log-linear parameters. *UAI*, 2013.

Shpitser, Evans, Richardson, and Robins – Introduction to Nested Markov Models. *Behaviormetrika*, 2014.

Spirtes, Glymour, Scheines – *Causation Prediction and Search*, 2nd Edition, MIT Press, 2000.

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Completeness

How do we know there isn't a 'third' axiom we could invoke?

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So if the latent variable model is correct[2], fitting the nested model is asymptotically equivalent fitting the LV model.

However, there are additional **inequality constraints**.

Potentially unsatisfactory as may not be a causal model corresponding to our inferred parameters.

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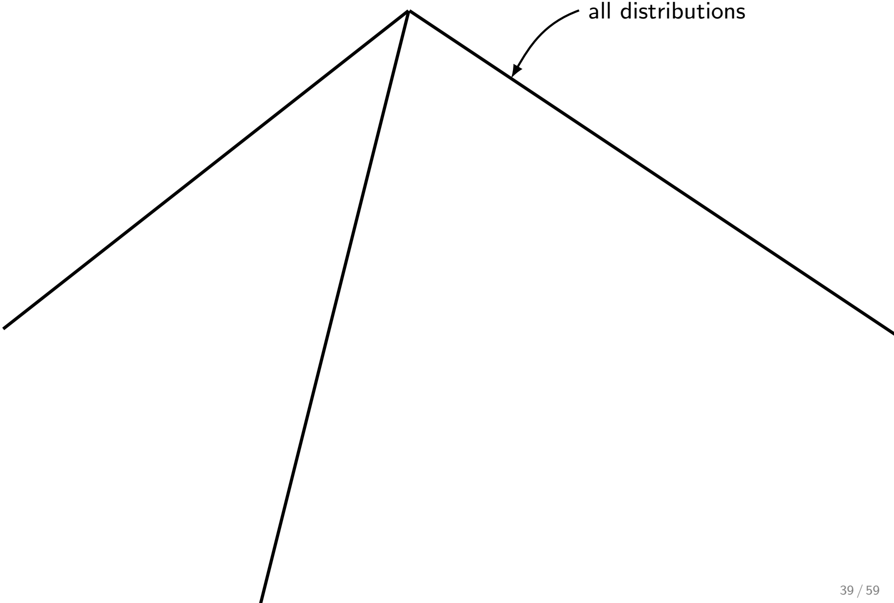
Getting the Picture



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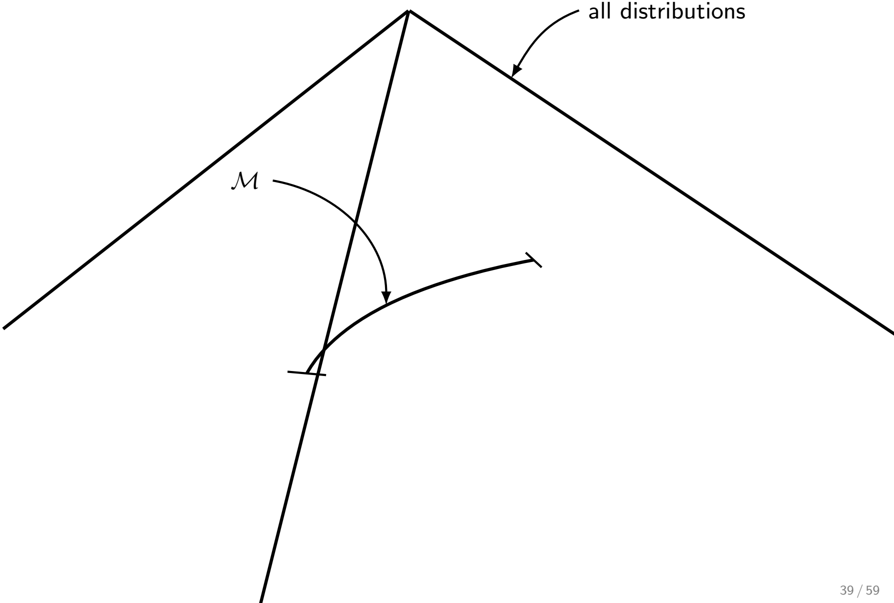

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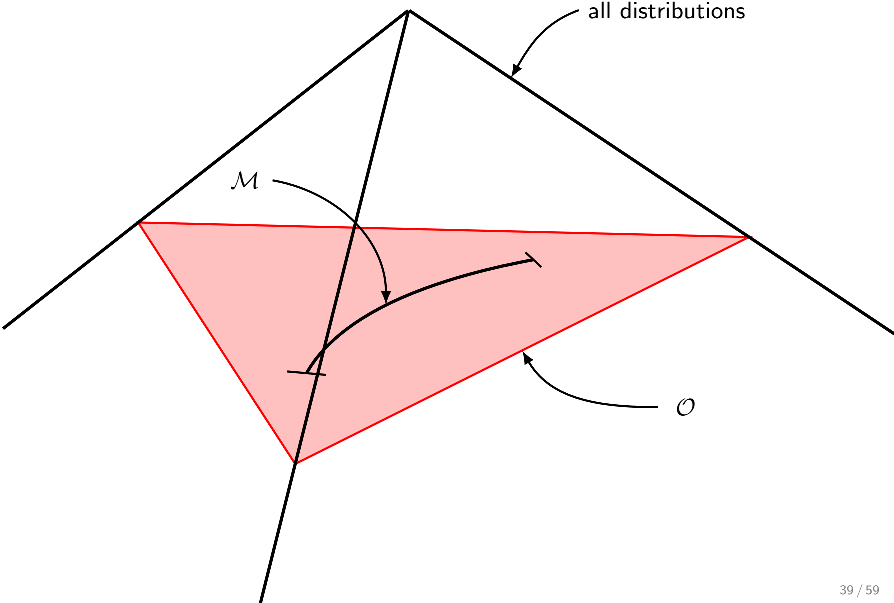

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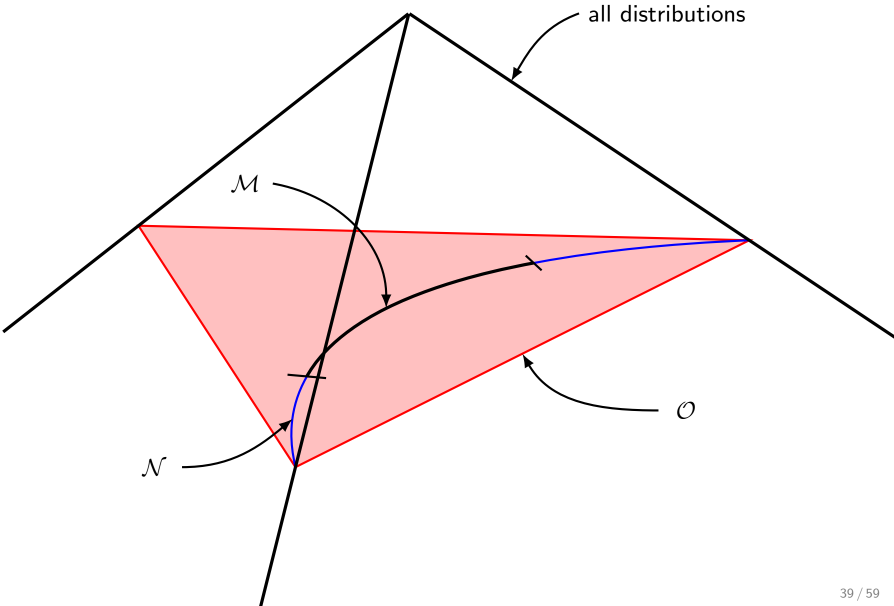

The IV Model

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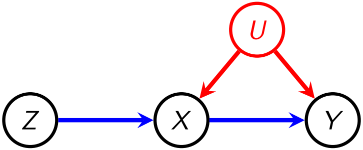

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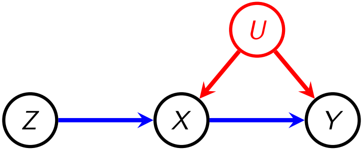

Marginalized DAG model

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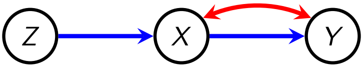

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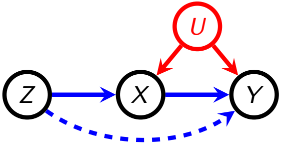
Nested Markov property gives saturated model, so true model of full dimension.

Instrumental Inequalities



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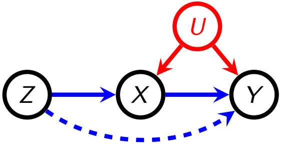

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$$\max_x \sum_y \max_z P(X = x, Y = y \mid Z = z) \leq 1. \qquad (*)$$

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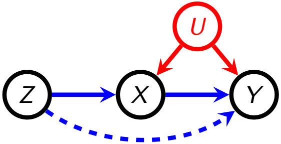

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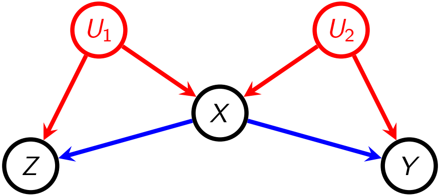
If Z, X, Y are binary, then (**??**) defines the marginalized DAG model (Bonet, 2001). e.g.

$$P(X = x, Y = 0 \mid Z = 0) + P(X = x, Y = 1 \mid Z = 1) \leq 1$$

The Problem

Question

How can we determine if a general marginalized DAG model induces inequality constraints?



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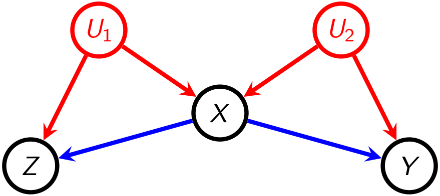

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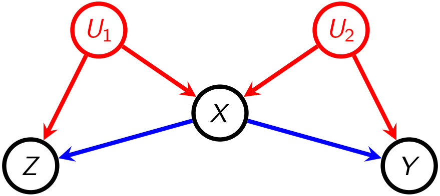

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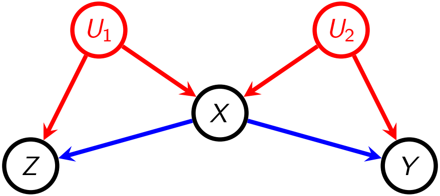

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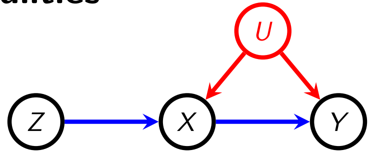
Computational linear algebra only works without adjacent latent variables. Also very computationally intensive.

Finding complete bounds in general is currently intractably hard.

Derivation of Inequalities



Have: $$p(x, y \mid z) = \int p(u)\, p(x \mid z, u)\, p(y \mid x, u)\, du.$$

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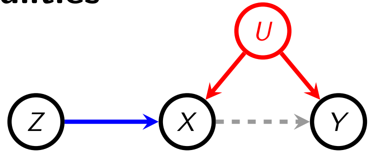

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Construct a **fictitious distribution** p^*:

$$p^*(x, y \mid z) = \int p(u)\, p(x \mid z, u)\, p(y \mid x = 0, u)\, du.$$

Now Y behaves as though $X = 0$ regardless of X's actual value.

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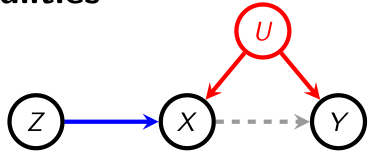

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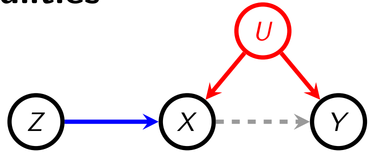

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Can't observe p^* but:

- **Consistency:** $p(0, y \mid z) = p^*(0, y \mid z)$ for each z, y; and
- **Independence:** $Y \perp\!\!\!\perp Z$ under p^*.

Derivation of Inequalities

For each $x = \xi$ we require p^*_ξ:

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We say that the probabilities $p(x, y \mid z)$ are **compatible** with $Y \perp\!\!\!\perp Z$.

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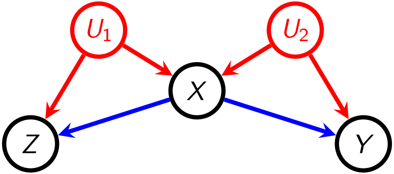

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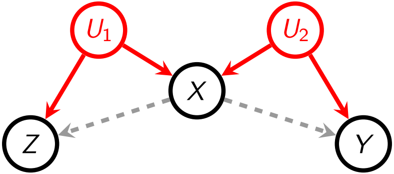

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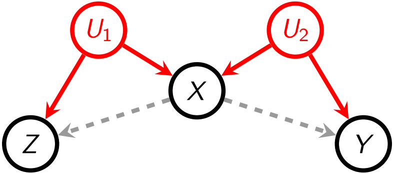

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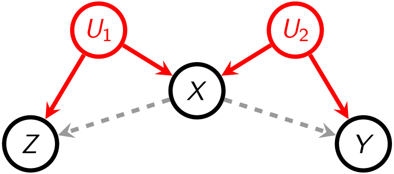

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[Note for the IV model, the conditional distribution $p(\xi, y \mid z)$ had to be compatible.]

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Most likely to happen if $p(x)$ is large for some value of x.

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If, in addition, $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$, $\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ and $\boldsymbol{X}_2, \boldsymbol{Y}_2$ are not descendants of $\boldsymbol{W}$, then

$$p(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{\omega} \mid \boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{z}) \qquad \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{z}.$$

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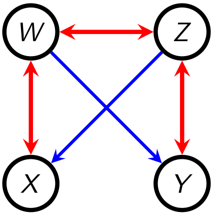
Whereas...

Corollary

If X and Y are not joined by an edge, nor share a hidden common cause, then a constraint is always induced on the joint distribution.

Example 1



X and Y cannot be d-separated in this graph $\implies$ no independences.

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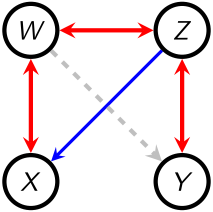

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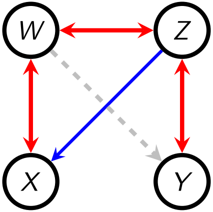

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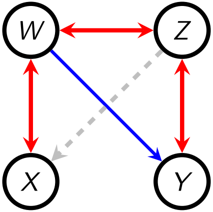

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By symmetry: $p(y, z \mid w, x)$ compatible with $X \perp\!\!\!\perp Y \mid W$ for each z.

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In general, Theorem 1 gives necessary but not sufficient conditions for p to be in the marginalized DAG model.

Equivalence on Three Variables

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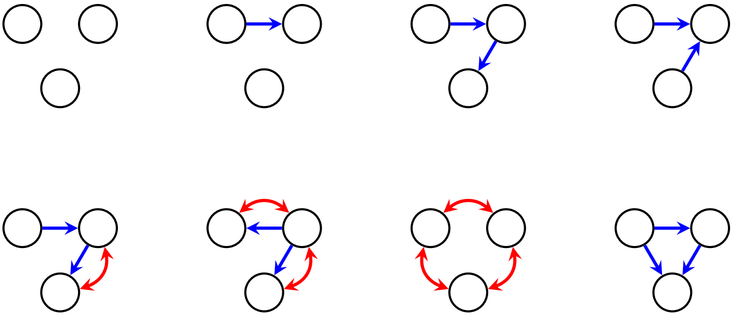
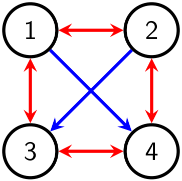
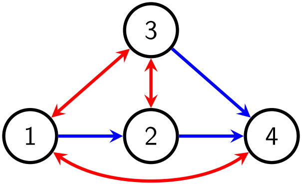
Even Markov equivalence is hard. Using Evans (2014), find 8 Markov equivalence classes on three variables.

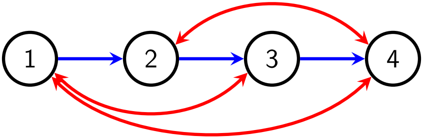


But Not on Four!

On four variables, it's still not clear whether or not the following models are saturated: (they are of full dimension in the discrete case)

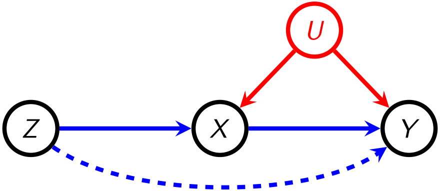






Outline

Causal Effects



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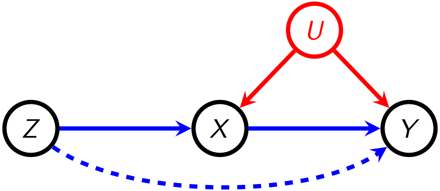

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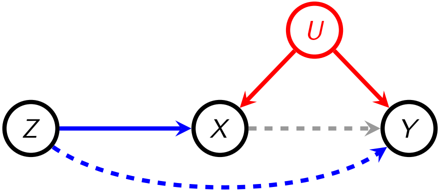

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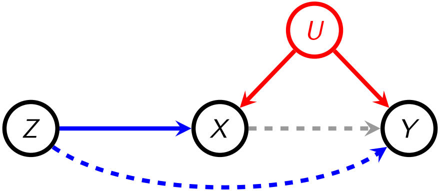

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$$\begin{aligned} p(y \mid \mathrm{do}(x = \xi, z)) &= p^*_\xi(y \mid z) \\ &= p(x, y \mid z) + \sum_{x' \neq \xi} p^*_\xi(x', y \mid z). \end{aligned}$$

Causal Bounds

This approach gives bounds on the interventional distributions (Evans, 2012) and, for example, the **average controlled direct effect**

$$\mathrm{ACDE}_{Z \to Y}(x) \equiv p(y = 1 \mid \mathrm{do}(x, z = 1)) - p(y = 1 \mid \mathrm{do}(x, z = 0)).$$

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Let $X \to Y$, but otherwise d-separated in the graph $\mathcal{G}^{\underline{\boldsymbol{W}}}$. Then an upper-bound on $\mathrm{ACDE}_{X \to Y}(w)$ is given by maximizing

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This is just a quadratic equation. There is an analogous lower-bound.

Bounds: Special Case

Theorem

Let $X \to Y$, but otherwise d-separated in the graph $\mathcal{G}^{\underline{\boldsymbol{W}}}$, and that X is not a descendant of any variable in $\boldsymbol{W}$. Then

$$p(y=0,\boldsymbol{\omega} \,|\, x=0) + p(y=1,\boldsymbol{\omega} \,|\, x=1) - 1 \\ \leq \mathrm{ACDE}(\boldsymbol{\omega}) \leq \\ 1 - p(y=0,\boldsymbol{\omega} \,|\, x=1) - p(y=1,\boldsymbol{\omega} \,|\, x=0).$$

For the IV model, this is the tight bound given by Cai et al (2008).

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If bounds exclude zero then models violate Theorem 1 compatibility.

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- Complete inequality constraints seem very complicated (though some hope exists).
- Performing inference for inequality constraints with finite samples is non-trivial.
- Not obvious how to integrate inequalities into the previous parameterization.

Inequality References

Bonet – Instrumentality tests revisited, *UAI*, 2001.

Cai, Kuroki, Pearl and Tian – Bounds on direct effects in the presence of confounded intermediate variables, *Biometrics*, 64(3):695–701, 2008.

Evans – Graphical methods for inequality constraints in marginalized DAGs, *MLSP*, 2012.

Evans – Margins of discrete Bayesian networks, *arXiv:1501.02103*, 2015.

Kang and Tian – Inequality Constraints in Causal Models with Hidden Variables, *UAI*, 2006.

Pearl – On the testability of causal models with latent and instrumental variables, *UAI*, 1995.

Partition Function for General Sets

Let $\mathcal{I}(\mathcal{G})$ be the intrinsic sets of $\mathcal{G}$. Define a partial ordering $\prec$ on $\mathcal{I}(\mathcal{G})$ by $S_1 \prec S_2$ if and only if $S_1 \subset S_2$. This induces an isomorphic partial ordering on the corresponding recursive heads.

For any $B \subseteq V$ let

$$\Phi_{\mathcal{G}}(B) = \{H \subseteq B \,|\, H \text{ maximal under } \prec \text{ among heads contained in } B\};$$
$$\phi_{\mathcal{G}}(B) = \bigcup_{H \in \Phi_{\mathcal{G}}(B)} H.$$

So $\Phi_{\mathcal{G}}(B)$ is the 'maximal heads' in B, $\phi_{\mathcal{G}}(B)$ is their union.

Define (recursively)

$$[\emptyset]_{\mathcal{G}} \equiv \emptyset$$
$$[B]_{\mathcal{G}} \equiv \Phi_{\mathcal{G}}(B) \cup [\phi_{\mathcal{G}}(B)]_{\mathcal{G}}.$$

Then $[B]_{\mathcal{G}}$ is a partition of B.

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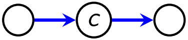

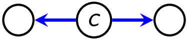

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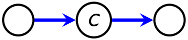

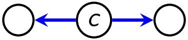

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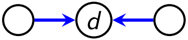

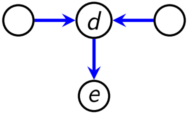

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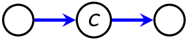

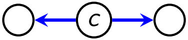

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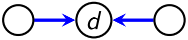

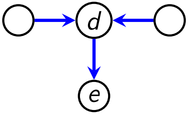

Two vertices v and w are **d-separated** given $C \subseteq V \setminus \{v, w\}$ if **all** paths are blocked.