Causal Inference and Graphical Models - II

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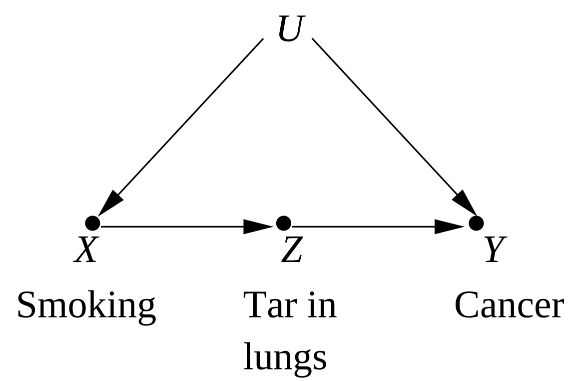
Outline

- Computing the effects of manipulations
- Inferring constraints implied by DAGs with hidden variables
  - on nonexperimental data
  - on experimental data
- Determining the causes of effects
  - Counterfactuals
  - Probabilities of causation

Causal Bayesian Networks

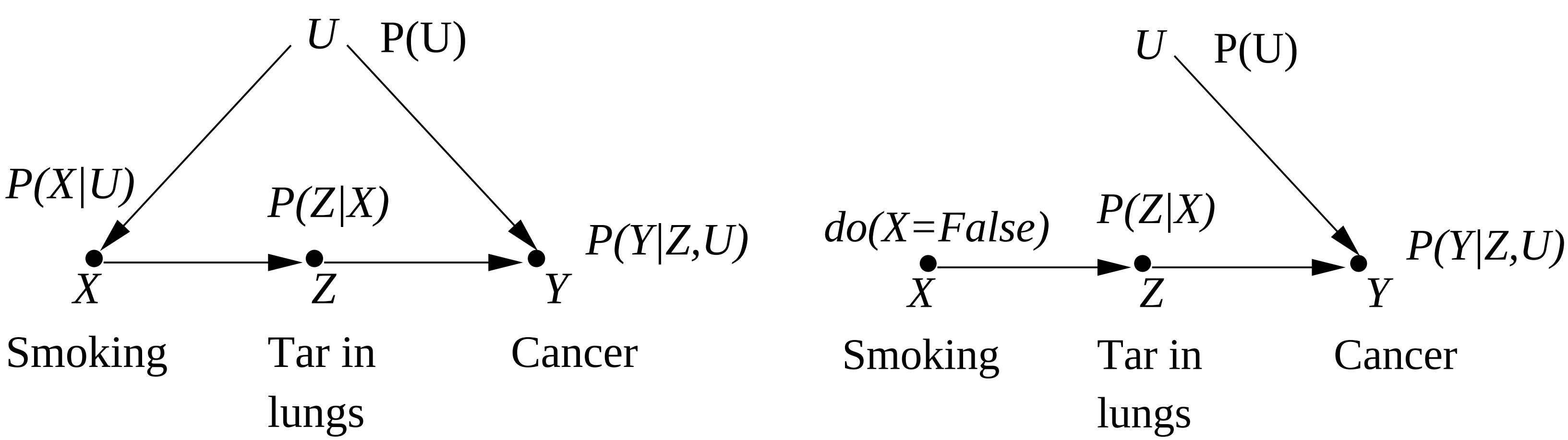
- **Causal graph**, a DAG,
  - Nodes: random variables.
  - Edges: direct causal influence.



- **Modularity**: Each parent-child relationship represents an autonomous causal mechanism.
  - Functional: $v_i = f(pa_i, \varepsilon)$
  - Probabilistic: $P(v_i | pa_i)$

Atomic Intervention/Manipulation

- $do(T=t)$: fixing a set T of variables to some constants $T=t$.



$$P(u,x,z,y) = P(u)P(x|u)P(z|x)P(y|z,u)$$

$$P_{X=False}(u,z,y) = P(u)P(z|X=False)P(y|z,u)$$

Terminologies and Notations

- Effects of manipulations/interventions/actions
- The **causal effect** of T on S: $P_t(s)$.
- Notations:

$$P_t(s) = P(s|do(t)) = P(s|set(t)) = P(s|\hat{t}\,) = P(s||t)$$

Computing Causal Effects

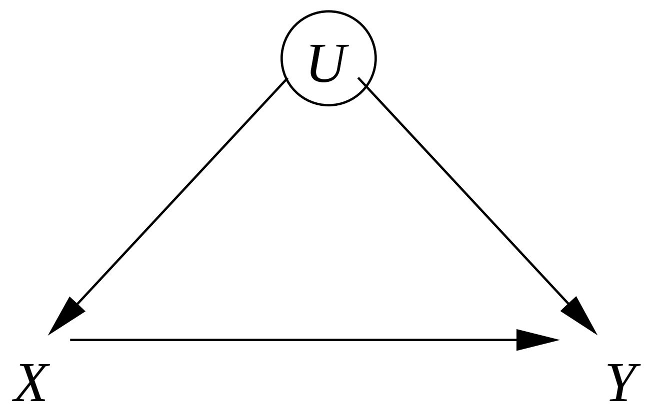
- Given:
  - observational data: distribution $P(v)$
  - qualitative causal assumptions: a causal graph
- Can we compute the causal effect $P_t(s)$.
- Causal BNs with no hidden common causes

$$P(v) = \prod_i P(v_i|pa_i)$$

$$P_t(v) = \prod_{\{i|V_i \notin T\}} P(v_i|pa_i)$$

Computing Causal Effects

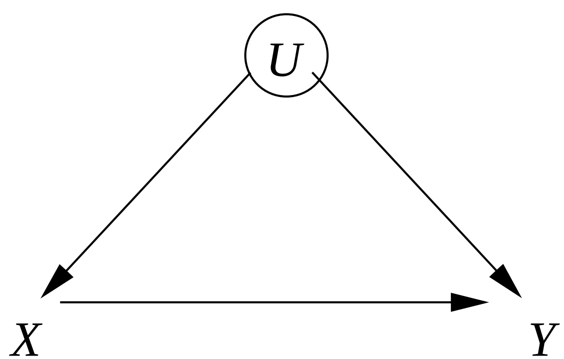
- The presence of unobserved (hidden, latent) variables.



Input: causal graph $+ P(x,y)$.
Can we predict $P_x(y)$?

Computing Causal Effects

- Unidentifiable



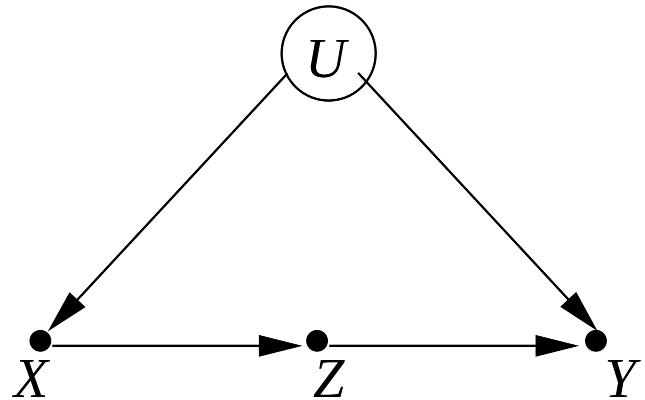
$$P(x,y) = \sum_u P^{M_1}(x|u)P^{M_1}(y|x,u)P^{M_1}(u)$$
$$= \sum_u P^{M_2}(x|u)P^{M_2}(y|x,u)P^{M_2}(u)$$

$$P_x^{M_1}(y) = \sum_u P^{M_1}(y|x,u)P^{M_1}(u)$$

$$P_x^{M_2}(y) = \sum_u P^{M_2}(y|x,u)P^{M_2}(u)$$

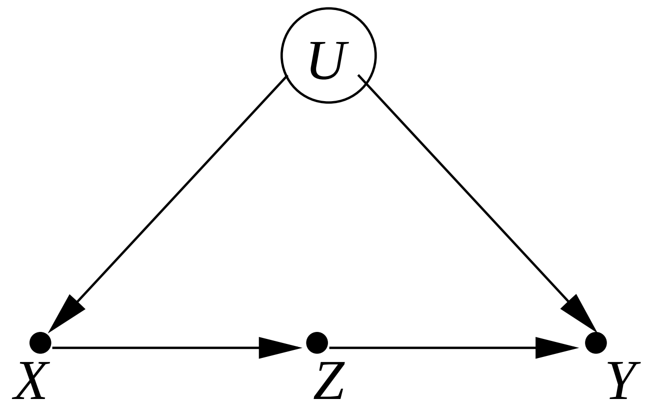
$$P_x^{M_1}(y) \neq P_x^{M_2}(y)$$

Computing Causal Effects



Input: causal graph + $P(x, y, z)$.

Computing Causal Effects



Input: causal graph $+ P(x, y, z)$.

Output:

$$P_x(y) = \sum_z P(z|x) \sum_{x'} P(y|x', z) P(x')$$

- Identifiable

Causal Calculus

- Pearl's *do*-calculus

Rule 1: Ignoring observations

$$P_x(y|z,w) = P_x(y|w) \quad \text{if } (Y \perp\!\!\!\perp Z | X, W)_{G_{\overline{X}}}$$

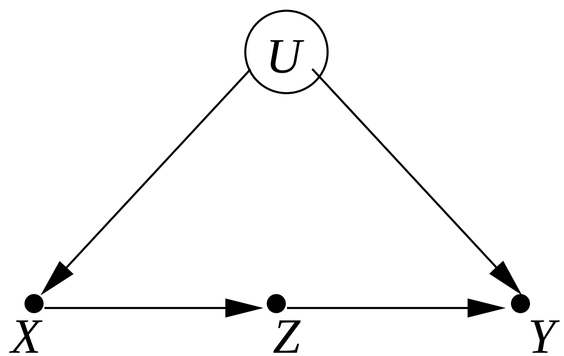
Rule 2: Action/observation exchange

$$P_{x,z}(y|w) = P_x(y|z,w) \quad \text{if } (Y \perp\!\!\!\perp Z | X, W)_{G_{\overline{X}\underline{Z}}}$$

Rule 3: Ignoring actions

$$P_{x,z}(y|w) = P_x(y|w) \quad \text{if } (Y \perp\!\!\!\perp Z | X, W)_{G_{\overline{X}\overline{Z(W)}}}$$

Computing In Do-calculus

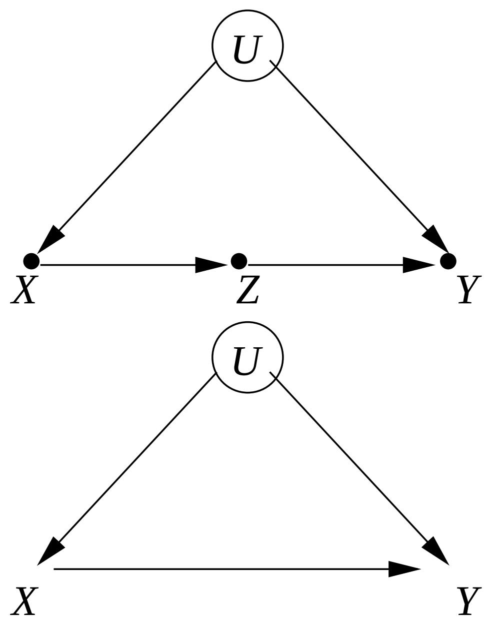


$$
\begin{aligned}
P_x(y) &= \sum_z P_x(y|z)P_x(z) \\
&= \sum_z P_x(y|z)P(z|x) \quad \text{Rule 2} \\
&= \sum_z P_{x,z}(y)P(z|x) \quad \text{Rule 2} \\
&= \sum_z P_z(y)P(z|x) \quad \text{Rule 3} \\
&= \sum_z \sum_{x'} P_z(y|x')P_z(x')P(z|x) = \dots \\
&= \sum_z P(z|x) \sum_{x'} P(y|x',z)P(x')
\end{aligned}
$$

- When to use which rule of *do*-calculus?

Semi-Markovian Models

- For convenience of presentation, consider models in which each hidden variable is a root node and has exactly two observed children.



Semi-Markovian Models

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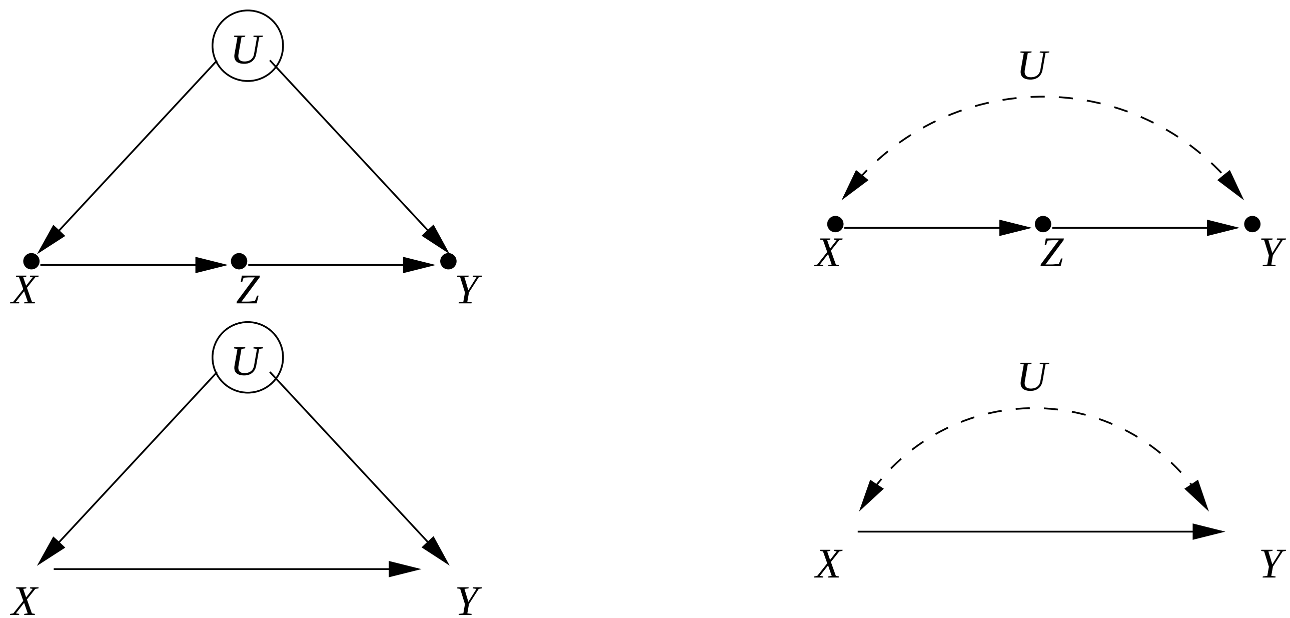
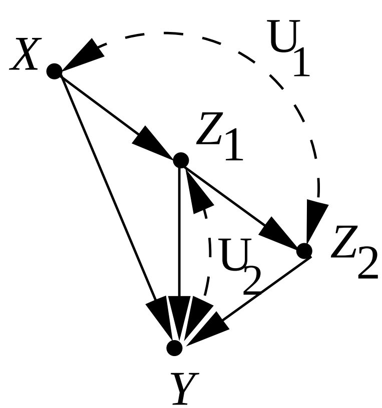


- Represent the presence of hidden variables with bidirected links.

C-components

- Variables are partitioned into c-components.
- Two variables are in the same c-components iff they are connected by a bi-directed path.
- Bi-directed path: each link on the path is a bidirected link.



Two c-components:

$S_1 = \{X, Z_2\}$

$S_2 = \{Z_1, Y\}$

Decomposition of $P(v)$

$$P(v) = \sum_u \prod_{\{i|V_i \in V\}} P(v_i|pa_{v_i}) \prod_{\{i|U_i \in U\}} P(u_i)$$

For any set $S \subseteq V$, define

$$Q[S](v) = P_{v \setminus s}(s) = \sum_u \prod_{\{i|V_i \in S\}} P(v_i|pa_{v_i}) \prod_{\{i|U_i \in U\}} P(u_i)$$

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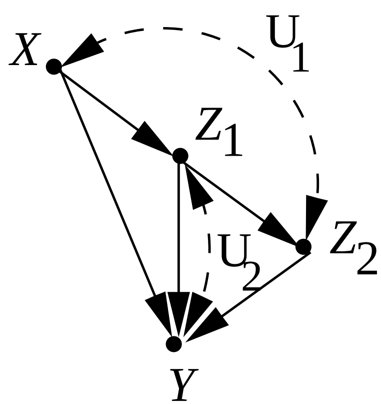
Theorem (Decomposition of joint) *Let a causal graph be partitioned into c-components $S_1, \ldots, S_k$. Then*

$$P(v) = \prod_i Q[S_i](v) = \prod_i P_{v \setminus s_i}(s_i)$$

Decomposition of $P(v)$



$$P(x,y,z_1,z_2)$$
$$= \sum_{u_1,u_2} P(x|u_1)P(z_1|x,u_2)P(z_2|z_1,u_1)$$
$$P(y|x,z_1,z_2,u_2)P(u_1)P(u_2)$$

Two c-components:

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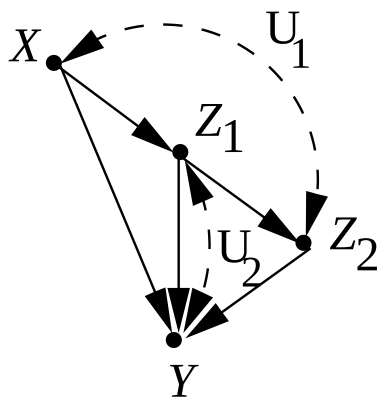
Decomposition of $P(v)$



Two c-components:

$S_1 = \{X, Z_2\}$

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$$
\begin{aligned}
&P(x,y,z_1,z_2)\\
&= \sum_{u_1,u_2} P(x|u_1)P(z_1|x,u_2)P(z_2|z_1,u_1)\\
&\qquad P(y|x,z_1,z_2,u_2)P(u_1)P(u_2)\\
&= \left(\sum_{u_1} P(x|u_1)P(z_2|z_1,u_1)P(u_1)\right)\\
&\quad \left(\sum_{u_2} P(z_1|x,u_2)P(y|x,z_1,z_2,u_2)P(u_2)\right)\\
&= Q[S_1](x,z_1,z_2)Q[S_2](x,z_1,z_2,y)\\
&= P_{y,z_1}(x,z_2)P_{x,z_2}(y,z_1)
\end{aligned}
$$

Computing $Q[S_i]$'s

Theorem *Let a causal graph be partitioned into c-components $S_1, \ldots, S_k$. Then each $Q[S_i]$ is identifiable and is given by*

$$Q[S_i](v) = P_{v \setminus s_i}(s_i) = \prod_{\{j | V_j \in S_i\}} P(v_j | v_1, \ldots, v_{j-1}),$$

assuming a topological order over V be $V_1 < \ldots < V_n$.

Conditional Independences

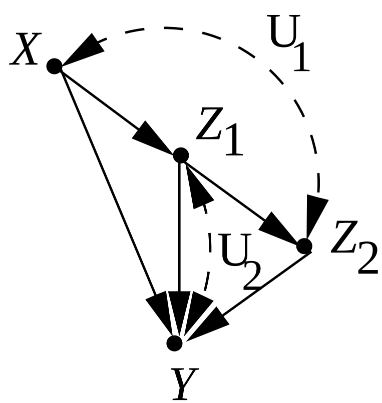
Theorem *Let a topological order over V be $V_1 < \ldots < V_n$,*

$$P(v_i|v_1,\ldots,v_{i-1}) = P(v_i|pa(T_i)\setminus\{v_i\})$$

where T_i is the c-component of the subgraph $G_{\{V_1,\ldots,V_i\}}$ that contains V_i.

- In the presence of hidden variables, each variable is independent of its non-descendants given its parents, the non-descendant variables in its c-component, and the parents of the non-descendant variables in its c-component.

An Example



Two c-components:

$S_1 = \{X, Z_2\}$

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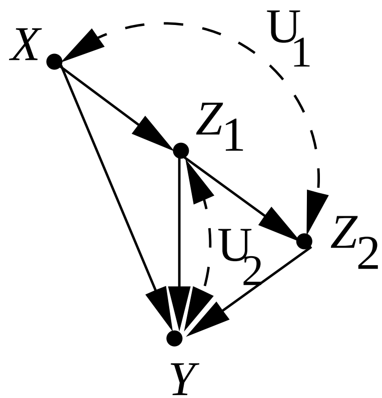
Topological order:

$X < Z_1 < Z_2 < Y$

An Example



Two c-components:

$S_1 = \{X, Z_2\}$

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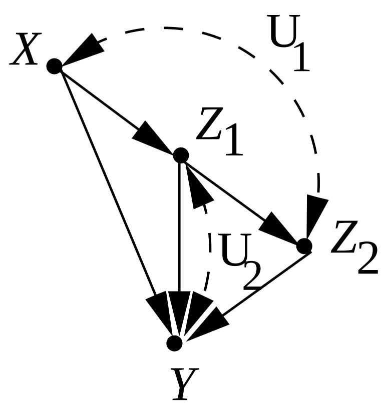
Topological order:

$X < Z_1 < Z_2 < Y$

$$P(x, y, z_1, z_2) = Q[\{X, Z_2\}]Q[\{Z_1, Y\}]$$

An Example



Two c-components:

$S_1 = \{X, Z_2\}$

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Topological order:

$X < Z_1 < Z_2 < Y$

$$P(x, y, z_1, z_2) = Q[\{X, Z_2\}]Q[\{Z_1, Y\}]$$

$$Q[\{X, Z_2\}] = P_{y,z_1}(x, z_2) = P(x)P(z_2|x, z_1)$$

$$Q[\{Z_1, Y\}] = P_{x,z_2}(y, z_1) = P(z_1|x)P(y|x, z_1, z_2)$$

Decomposition of $P_{v\backslash h}(h)$

Theorem Let $H \subseteq V$, and G_H denote the subgraph of G composed only of the variables in H. Assume G_H is partitioned into c-components $H_1, \ldots, H_l$. Then

1. $$Q[H] = \prod_i Q[H_i], \quad i.e., \quad P_{v\backslash h}(h) = \prod_i P_{v\backslash h_i}(h_i).$$
2. Each $Q[H_i] = P_{v\backslash h_i}(h_i)$ is computable in terms of $Q[H] = P_{v\backslash h}(h)$.

Computing $Q[S]$

A procedure for computing $Q[S](v) = P_{v\setminus s}(s)$ is developed, that

1. Determine the identifiability of $Q[S]$.
2. Express identifiable $Q[S]$ in terms of $P(v)$.

Identifying Causal Effects $P_t(s)$

Let $D = An(S)_{G_{V\setminus T}}$, and assume that the subgraph G_D is partitioned into c-components $D_1, \ldots, D_k$. Then

$$
\begin{aligned}
P_t(s) &= \sum_{(v\setminus t)\setminus s} P_t(v \setminus t) \\
&= \sum_{(v\setminus t)\setminus s} Q[V \setminus T] \\
&\ldots \\
&= \sum_{d\setminus s} \prod_i Q[D_i].
\end{aligned}
$$

- $P_t(s)$ is identifiable iff each $Q[D_i]$ is identifiable.

Computing $P_t(s)$ – Summary

- **A complete algorithm** is developed that will either determine $P_t(s)$ to be unidentifiable or express $P_t(s)$ in terms of $P(v)$
- **Do-calculus is complete** for computing causal effects
- Open questions:
  - computing causal effects in partially known DAGs, or PAGs

Outline

- Computing the effects of manipulations
- **Inferring constraints implied by DAGs with hidden variables**
- Determining the causes of effects

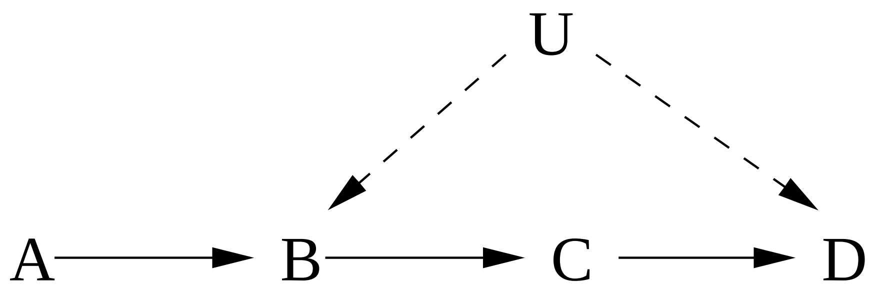
Implications of Causal Models

- The validity of a causal model can be tested only if it has empirical implications, that is, it must impose constraints on data.
- No hidden variables:
  - observational implications of a BN are completely captured by conditional independence relationships
  - read by d-separation

Implications of Causal Models

- The validity of a causal model can be tested only if it has empirical implications, that is, it must impose constraints on data.
- No hidden variables:
  - observational implications of a BN are completely captured by conditional independence relationships
  - read by d-separation
- When hidden variables are present:
  - other types of constraints on the observed distribution.

An Example



- $P(a,b,c,d)$ must satisfy:

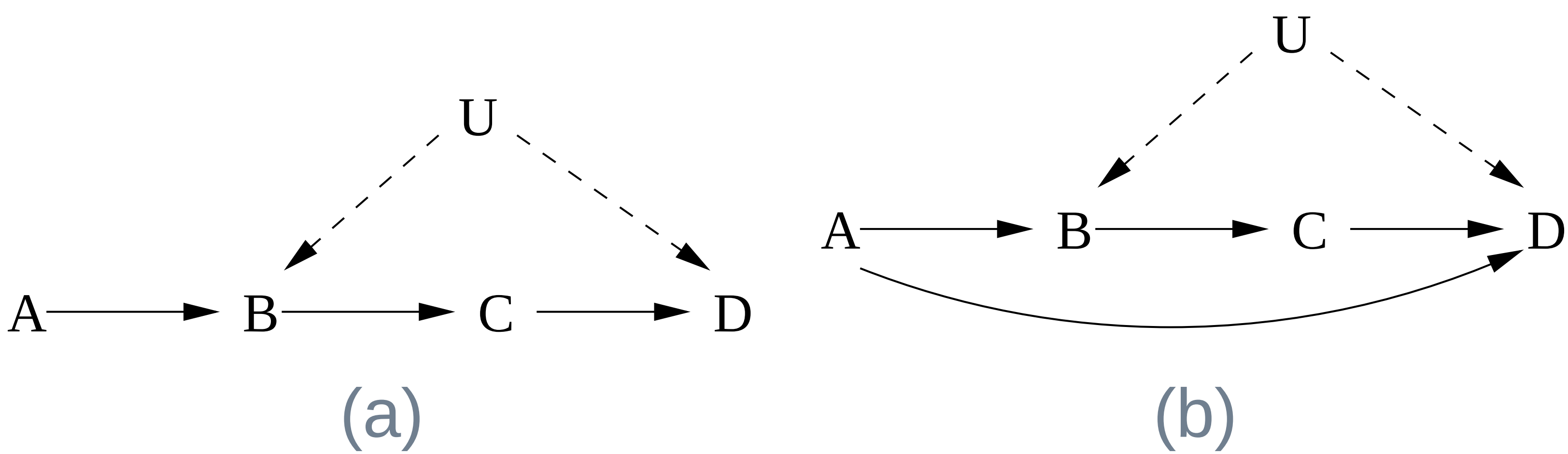
$$\sum_b P(d|a,b,c)P(b|a) = f(c,d)$$

$$i.e. \quad \sum_b P(d|a,b,c)P(b|a) = \sum_b P(d|a',b,c)P(b|a')$$

- Functional constraints

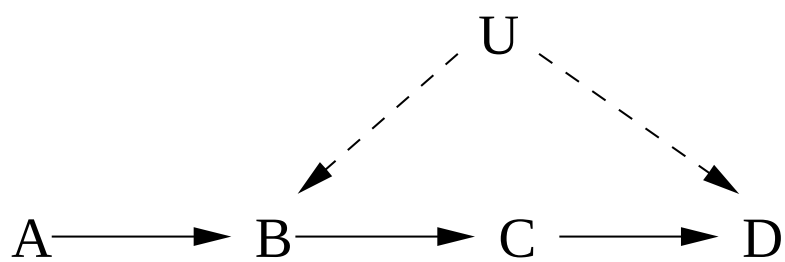
Applications

- Empirically validating causal models.
- Distinguishing causal models with the same set of conditional independence relationships.



Independence statements: A is independent of C given B.

Inferring Functional Constraints



- Consider

$$Q[\{D\}] = P_{a,b,c}(d) = \sum_u P(d|c,u)P(u) \equiv Q[\{D\}](c,d)$$

- $Q[\{D\}]$ is identifiable as

$$Q[\{D\}](v) = \sum_b P(d|a,b,c)P(b|a).$$

- Therefore $\sum_b P(d|a,b,c)P(b|a)$ is independent of a.

Inferring Functional Constraints

Basic Ideas

- $Q[S](v)$ is a function of values only of a subset of V.
- Whenever $Q[S]$ is computable from $P(v)$, it may lead to some constraints — conditional independence relations or functional constraints.

The Arguments of $Q[S]$

$$Q[S](v) = \sum_u \prod_{\{i|V_i \in S\}} P(v_i|pa_{v_i}) \prod_{\{i|U_i \in U\}} P(u_i)$$

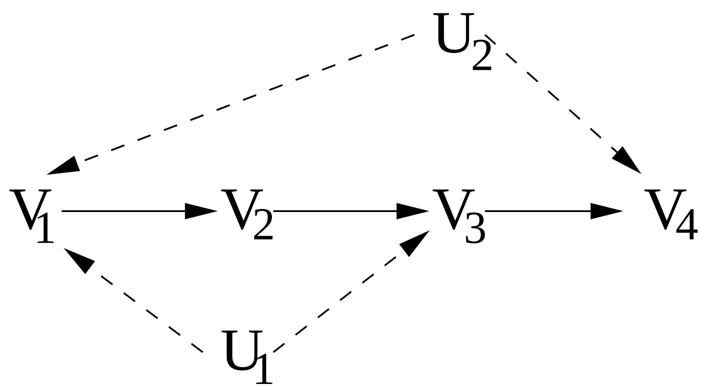
- $Pa(S)$: the union of S and the set of parents of S.
- $Q[S](v)$ *is a function of* $Pa(S)$:

$$Q[S](v) = Q[S](pa(S))$$

Identifying Functional Constraints

1. Find a computable $Q[S]$ expressed in terms of $P(v)$
   - A procedure is developed that systematically find computable $Q[S]$.
2. $Q[S]$ is a function only of $pa(S)$ $\Longrightarrow$ conditional independence relations or functional constraints.

Another Example



- The model does not imply any conditional independences

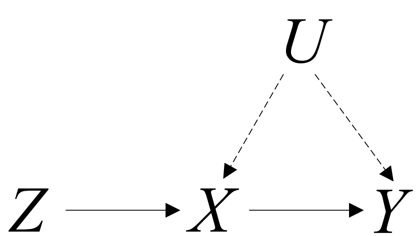
$$Q[\{V_4\}](v_3, v_4) = \frac{\sum_{v_1} P(v_4|v_3, v_2, v_1)P(v_3|v_2, v_1)P(v_1)}{\sum_{v_1} P(v_3|v_2, v_1)P(v_1)}.$$

- The right hand side is independent of v_2.

Inequality Constraints

- Pearl's *instrumental inequality*, for discrete variables

$$\max_x \sum_y [\max_z P(xy|z)] \leq 1.$$



E.g., binary variables

$$P(x_0,y_0|z_0) + P(x_0,y_1|z_1) \leq 1$$

$$P(x_1,y_0|z_0) + P(x_1,y_1|z_1) \leq 1$$

$$P(x_0,y_1|z_0) + P(x_0,y_0|z_1) \leq 1$$

$$P(x_1,y_1|z_0) + P(x_1,y_0|z_1) \leq 1$$

Inequality Constraints

- Empirically validating causal models.
- Distinguishing causal models with the same set of conditional independence relationships.
- Open problem: how to identify inequality constraints

Constraints on Experimental Data

- A causal BN not only imposes constraints on the nonexperimental distribution but also on the experimental distributions
- A causal BN can be tested and falsified by using two types of data:
  - nonexperimental data are passively observed,
  - experimental data are produced by manipulating (randomly) some variables and observing the states of other variables.
- The ability to use a mixture of nonexperimental and experimental data will greatly increase our power of causal reasoning and learning.

Constraints on Experimental Data

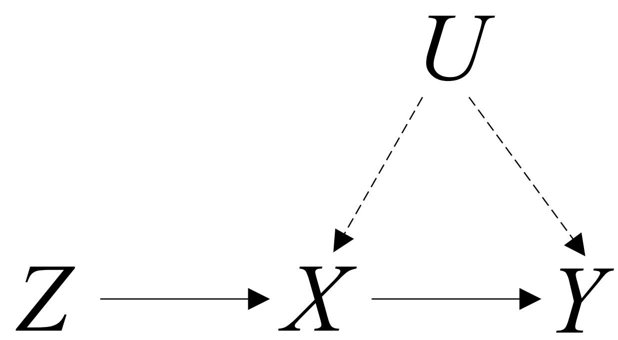
- Let $H \subseteq V$ and assume the subgraph G_H is partitioned into c-components $H_1, \ldots, H_l$. Then

$$P_{v\backslash h}(h) = \prod_i P_{v\backslash h_i}(h_i).$$

- $$P_{pa_i,s}(v_i) = P_{pa_i}(v_i), \;\; \forall S \subseteq V \setminus (PA_i \cup \{V_i\})$$
- *If a set T is composed of nondescendants of V_j,*

$$P_{v_j,s}(t) = P_s(t).$$

Constraints on Experimental Data



$$P_z(xy) = P(xy|z)$$

$$P_{yz}(x) = P(x|z)$$

$$P_{xz}(y) = P_x(y)$$

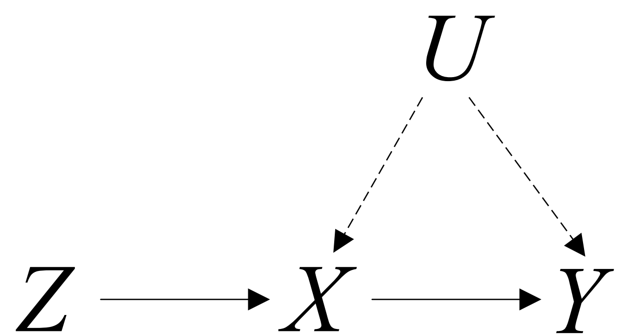
Inequalities on Experimental Data

- Consider discrete random variables
- A type of inequality constraints on experimental distributions
  - Let V be partitioned into c-components $T_1, \ldots, T_k$. For $i = 1, \ldots, k$, $\forall S_1 \subseteq T_i$,

$$\sum_{S_2 \subseteq T_i \setminus S_1} (-1)^{|S_2|} P_{v \setminus (s_1 \cup s_2)}(s_1, s_2) \geq 0, \quad \forall v \in Dm(V)$$

- Not complete

Inequalities on Experimental Data



For all $x \in Dm(X)$, $y \in Dm(Y)$, $z \in Dm(Z)$

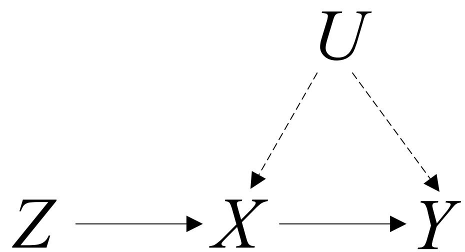
$$1 - P_{yz}(x) - P_{xz}(y) + P_z(xy) \geq 0$$
$$P_{yz}(x) - P_z(xy) \geq 0$$
$$P_{xz}(y) - P_z(xy) \geq 0$$

Applications of Inequalities

- Model testing using a mixture of nonexperimental and experimental data
- Bounding (unidentifiable) causal effects from nonexperimental data
- Bounding the effects of untried interventions from experiments involving auxiliary interventions that are easier or cheaper to implement

$$P_z(x,y) \leq P_{xz}(y) \leq 1 - P_z(x) + P_z(x,y)$$

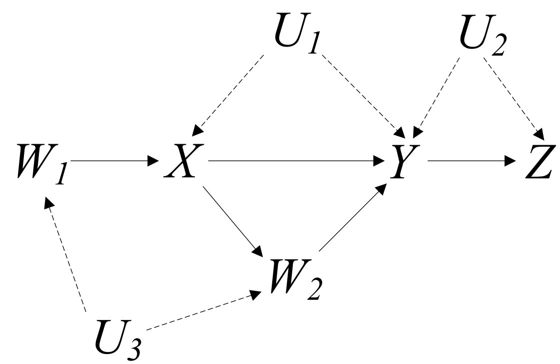
Deriving Instrumental Inequality



- Equality constraints: $P_z(xy) = P(xy|z)$, $P_{xz}(y) = P_x(y)$
- Inequality: $P_z(xy) \leq P_{xz}(y)$
- We have

$$P(xy|z) \leq P_x(y)$$
$$\max_z P(xy|z) \leq P_x(y)$$
$$\sum_y \max_z P(xy|z) \leq 1$$

Deriving Instrumental Inequality



- The following instrumental type inequality can be derived

$$\sum_{yz} \max_{w_1} P(z|w_1 x w_2 y) P(y|w_1 x w_2) P(x|w_1) \leq 1.$$

Experimental Implications

- What if causal structures unknown?
- Given a collection of experimental distributions

$$P_* = \{P_t(v) | T \subseteq V, t \in Dm(T)\}$$

  - Is the collection P_* compatible with some underlying causal Bayesian network?

Three Properties

- If no hidden variables

1. Effectiveness

$$P_t(t) = 1.$$

2. Markov

$$P_{v\setminus(s_1 \cup s_2)}(s_1, s_2) = P_{v\setminus s_1}(s_1) P_{v\setminus s_2}(s_2)$$

3. Recursiveness
   Define $X \rightsquigarrow Y$ as $\exists w$, $P_{x,w}(y) \neq P_w(y)$,

$$(X_0 \rightsquigarrow X_1) \wedge \ldots \wedge (X_{k-1} \rightsquigarrow X_k) \Rightarrow \neg(X_k \rightsquigarrow X_0)$$

A Complete Characterization

Theorem (Soundness) *Effectiveness, Markov, and recursiveness hold in all causal Bayesian networks.*

Theorem (Completeness) *If a P_* set satisfies effectiveness, Markov, and recursiveness, then there exists a causal Bayesian network with a unique causal graph that can generate this P_* set.*

Semi-Markovian Models

- Effectiveness
- Recursiveness
- Directionality
  There exists a total order "$<$" such that

$$P_{v_i,w}(s) = P_w(s) \quad \text{if } \forall X \in S, X < V_i,$$

- Inclusion-Exclusion Inequalities
  For any subset $S_1 \subseteq V$,

$$\sum_{S_2 \subseteq V \setminus S_1} (-1)^{|S_2|} P_{v \setminus (s_1 \cup s_2)}(v) \geq 0, \quad \forall v \in Dm(V),$$

A Complete Characterization

Theorem (Soundness) *Effectiveness, recursiveness, directionality, and inclusion-exclusion inequalities hold in all semi-Markovian models.*

Theorem (Completeness) *If a P_* set satisfies effectiveness, recursiveness, directionality, and inclusion-exclusion inequalities, then there exists a semi-Markovian model that can generate this P_* set.*

Applications of Characterization

- Reasoning about causal effects without possessing causal structures
- Is a collection of experimental distributions compatible?
- Predicting about or bounding interventions that were not tried experimentally even if the structure of the underlying model is unknown

Open Problems

- Identifying *all* constraints
  - on nonexperimental distributions
  - on experimental distributions
  - equalities
  - inequalities
  - constraints particular to a family of distributions
- Using constraints to guide learning BNs with hidden variables

Outline

- Computing the effects of manipulations
- Inferring constraints implied by DAGs with hidden variables
- **Determining the causes of effects**
  - Counterfactuals
  - Probabilities of causation

Determining the Causes of Effects

- Assessing the likelihood that one event *was the cause* of another
- Legal responsibility: Mr. A took a drug and died,
  - Lawsuit: the drug **caused** the death of Mr. A
  - Experimental and nonexperimental data on patients
  - Court to decide:
    Is it **more probable than not** that A would be alive **but for** the drug?

The Problem

- Probability of necessary causation (PN): "Probability that event y would not have occured if it were not for event x, given that x and y did in fact occur."
- What is the **meaning** of PN? How to define PN mathematically?
- Under what conditions can PN be **learned** from statistical data?

Functional Causal Models

- Structural Equations

$$v_i = f_i(pa_i, u_i), \qquad i = 1, \ldots, n.$$

- $U = \{U_1, \ldots, U_n\}$: exogenous background/error variables
- Acyclic models
  - The values of the V variables will be uniquely determined by those of the U variables.
  - The joint distribution $P(v)$ is determined uniquely by the distribution $P(u)$.
- $P(u)$ defines a probabilistic causal model

Counterfactuals

- An intervention is represented as an alteration on a select set of functions instead of a select set of conditional probabilities.
- The effect of $do(V_i = v_i)$ is represented by replacing the equation $v_i = f_i(pa_i, u_i)$ with

$$V_i = v_i$$

- The counterfactual expression “The value that Y would have obtained, had X been x”, denoted by $Y_x(u)$, is interpreted as the solution for Y in the modified set of equations in situation $U = u$.

Probabilities of Counterfactuals

$$P(Y = y) = \sum_{\{u \mid Y(u)=y\}} P(u)$$

$$P(Y_x = y) = \sum_{\{u \mid Y_x(u)=y\}} P(u) \equiv P_x(y)$$

$$P(Y_x = y, X = x') = \sum_{\{u \mid Y_x(u)=y \ \& \ X(u)=x'\}} P(u)$$

$$P(Y_x = y, Y_{x'} = y') = \sum_{\{u \mid Y_x(u)=y \ \& \ Y_{x'}(u)=y'\}} P(u)$$

Computing Counterfactuals

Given evidence $X = x', Y = y'$, compute the probability of $Y = y$ had X been x (X and Y subsets of variables):

Step 1 (***abduction***): Update the probability $P(u)$ to obtain $P(u|x', y')$.

Step 2 (***action***): Replace the equations corresponding to variables in set X by the equations $X = x$.

Step 3 (***prediction***): Use the modified model to compute the probability of $Y = y$.

Computing Counterfactuals

Model 1 $\quad x = u_1,$

$$y = u_2.$$

Model 2 $\quad x = u_1,$

$$y = xu_2 + (1-x)(1-u_2).$$

where U_1 and U_2 are two independent binary variables with $P(u_1 = 1) = P(u_2 = 1) = \frac{1}{2}$, leading to the same distribution $P(x,y)$.

Model 1: $P(Y_{x=0} = 0 | X = 1, Y = 1) = 0$

Model 2: $P(Y_{x=0} = 0 | X = 1, Y = 1) = 1$

Computing Counterfactuals

- Probabilistic causal models are insufficient for computing probabilities of counterfactuals; knowledge of the actual process behind $P(y|x)$ is needed for the computation.
- A functional causal model constitutes a mathematical object sufficient for the computation and definition of such probabilities.

Probabilities of Causation

- Let X and Y be two binary variables
- **Probability of necessity (PN)**

$$PN \equiv P(Y_{x'} = y' \mid X = x, Y = y) \equiv P(y'_{x'} | x, y)$$

- PN stands for the probability that event y would not have occurred in the absence of event x, $y'_{x'}$, given that x and y did in fact occur.
- Applications in epidemiology, legal reasoning, and AI: a certain case of disease is *attributable* to a particular exposure, "the probability that disease would not have occurred in the absence of exposure, given that disease and exposure did in fact occur."

Probabilities of Causation

- **Probability of sufficiency (PS)**

$$PS \equiv P(y_x | y', x')$$

- PS gives the probability that setting x would produce y in a situation where x and y are in fact absent.
- Applications in policy analysis, AI, and psychology: a policy maker interested in the dangers that a certain exposure may present to the healthy population, the "probability that a healthy unexposed individual would have gotten the disease had he/she been exposed."

Legal Responsibility

- A lawsuit is filed against the manufacturer of drug x, charging that the drug is likely to have caused the death of Mr. A, who took the drug to relieve symptom S associated with disease D
- Experimental and nonexperimental data (in the next page)
- Court to decide:
  Is it more probable than not that A would be alive but for the drug?
- Can PN be estimated from data?

Data for Legal Responsibility

Table 0: (Hypothetical) frequency data obtained in experimental and nonexperimental studies, comparing deaths (in thousands) among drug users, x, and non-users, x'.

	Experimental		Nonexperimental	
	x	x'	x	x'
Deaths(y)	16	14	2	28
Survivals(y')	984	986	998	972

LINEAR PROGRAMMING

- Parameters: $p_{110} = P(y_x, y_{x'}, x'), \ldots$
- Probabilistic constraints:

$$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} p_{ijk} = 1$$

$$p_{ijk} \geq 0 \text{ for } i,j,k \in \{0,1\}$$

- Nonexperimental constraints:

$$\begin{aligned} p_{111} + p_{101} &= P(x,y) \\ p_{011} + p_{001} &= P(x,y') \\ p_{110} + p_{010} &= P(x',y) \end{aligned}$$

Bounding by LP

- Experimental constraints:

$$
\begin{aligned}
P(y_x) &= p_{111} + p_{110} + p_{101} + p_{100} \\
P(y_{x'}) &= p_{111} + p_{110} + p_{011} + p_{010}
\end{aligned}
$$

- Maximize (Minimize)

$$
\begin{aligned}
PN &= p_{101}/P(x,y) \\
PS &= p_{100}/P(x',y')
\end{aligned}
$$

Typical Results

- Bounds on the probabilities of causation given combined nonexperimental and experimental data

$$\max\left\{\begin{array}{c} 0 \\ \frac{P(y)-P(y_{x'})}{P(x,y)} \end{array}\right\} \leq PN \leq \min\left\{\begin{array}{c} 1 \\ \frac{P(y'_{x'})-P(x',y')}{P(x,y)} \end{array}\right\}$$

$$\max\left\{\begin{array}{c} 0 \\ \frac{P(y_x)-P(y)}{P(x',y')} \end{array}\right\} \leq PS \leq \min\left\{\begin{array}{c} 1 \\ \frac{P(y_x)-P(x,y)}{P(x',y')} \end{array}\right\}$$

Solution to Legal Responsibility

- **Plaintiff:**

$$PN \geq \frac{P(y) - P(y_{x'})}{P(y, x)} = \frac{0.015 - 0.014}{0.001} = 1$$

- **Jury**: Guilty!

PERSONAL DECISION MAKING

Mr. B, survived without drug. Would he risk death by starting now?

- Nonexperimental data: $P(y|x) = 0.002$
- Experimental data: $P(y_x) = 0.016$
- Correct Answer: Risk = $PS = P(y_x|x', y')$

$$0.002 \leq PS \leq 0.031$$

Hierarchy of Causal Queries

- **Predictions** (conditioning) require only a specification of a joint distribution function.
- **Intervention** analysis requires a causal structure in addition to a joint distribution.
- **Counterfactual** analysis requires information about the functional relationships and the distribution of the omitted factors.